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Four-Flux and Warped Heterotic M-Theory Compactifications

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Abstract

In the framework of heterotic M-theory compactified on a Calabi-Yau threefold 'times' an interval, the relation between geometry and four-flux is derived *beyond first order*. Besides the case with general flux which cannot be described by a warped geometry one is naturally led to consider two special types of four-flux in detail. One choice shows how the M-theory relation between warped geometry and flux reproduces the analogous one of the weakly coupled heterotic string with torsion. The other one leads to a *quadratic* dependence of the Calabi-Yau volume with respect to the orbifold direction which avoids the problem with negative volume of the first order approximation. As in the first order analysis we still find that Newton's Constant is bounded from below at just the phenomenologically relevant value. However, the bound does not require an *ad hoc* truncation of the orbifold-size any longer. Finally we demonstrate explicitly that to leading order in $\kappa^{2/3}$ no Cosmological Constant is induced in the four-dimensional low-energy action. This is in accord with what one can expect from supersymmetry.

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1 Introduction and Summary

For heterotic M-theory compactified on a CY_3 the 4-form field-strength G does not vanish if higher order corrections in $\kappa^{2/3}$ are taken into account. The reason is that the boundary super Yang-Mills (SYM) theories represent magnetic sources which show up in the Bianchi-identity for G and require a G of order $\kappa^{2/3}$. Hence, by requiring SUSY which connects via the Killing-spinor equation metric quantities with the G -flux, we expect warped geometries to arise at this order, which is indeed the case [8]. An interesting interplay arises between the physics of G -fluxes and warped geometry (similar considerations for M- and F-theory on CY fourfolds can be found in [15],[16],[17],[18],[19],[20]; see also [21] for dynamical topology changes within heterotic M-theory). An important phenomenological issue is related to the value of Newton's Constant G_N . From dimensional reduction of heterotic M-theory on a CY_3 with volume $V(x^{11})$ (in the 11-dimensional metric), one can infer [8] (note that these expressions will become more refined if a non-trivial dependence on internal coordinates is kept, as will be explored in section 5 below)

$$G_N = \frac{\kappa^2}{16\pi\langle V \rangle d}, \quad \alpha_i = \frac{(4\pi\kappa^2)^{2/3}}{2V_i}, \quad (1.1)$$

where α_i is the gauge-coupling of the two ($i = 1, 2$) boundary E_8 SYM theories separated by a distance d and $V_1 = V(0), V_2 = V(d)$. Because G_N is related to gravity in the bulk, we use for its determination a bulk-averaged volume $\langle V \rangle = \frac{1}{d} \int_0^d dx^{11} V(x^{11})$. A determination of the warped geometry allows to calculate $V(x^{11})$ and thereby α_i and G_N . This had been undertaken in [8] to linearized order (first order in $\kappa^{2/3}$) with the result that $V(x^{11}) = -ax^{11} + V_0$, where the slope $a = \frac{1}{4\sqrt{2}} \int_{CY_3} d^6x \sqrt{g} \omega^{lm} \omega^{np} G_{lmnp} > 0$ is controlled by the G -flux. Here ω_{lm} denotes the Kähler-form on CY_3 . The surprising observation [8] has been that when the linear function $V(x^{11})$ becomes zero, the corresponding distance for d just gives rise to the correct value for G_N , whereas generically in heterotic string compactifications G_N is predicted too large by a factor of 400. Placing the second boundary at that distance means $\alpha_2 \rightarrow \infty$. Hence, the SYM there becomes strongly coupled and instanton contributions become relevant, which is the reason why this second boundary corresponds to a “hidden” world rather than our “observable” world. The first boundary at $x^{11} = 0$ instead allows for a perturbative SYM on it if V_1 is chosen huge enough such that $\alpha_1 \ll 1$ and consequently can be regarded as our “observable” world.

In this context some questions arise

- How is the linear behaviour of $V(x^{11})$, which leads to an unphysical negative volume

beyond a certain distance, changed in the full theory, i.e. beyond the leading $\kappa^{2/3}$ order?

- Does $V(x^{11})$ still keeps its attractive feature of becoming zero just at the phenomenological relevant distance? One has to recall that actually the linear approximation breaks down at the position of the zero. Is Newton's Constant still bounded from below beyond leading order?
- Does the phenomenologically relevant distance d get stabilized by an effective potential ?
- What is the relation of the compactification of heterotic M-theory with G -flux to the weakly coupled heterotic string with torsion [3] ?

The trouble with the linear behaviour is the following. Though one expects that quantum corrections will shift the actual value of $V(x^{11})$ slightly, small distortions of a linear function can never lift its zero – they can only shift its x^{11} position slightly, but the zero remains, as does the problem with the unphysical negative volume. Therefore it is important to determine the warp-factors and thereby $V(x^{11})$ beyond the leading order in $\kappa^{2/3}$, which we will undertake in this paper.

It may sound surprising how a result beyond order $\kappa^{2/3}$ can be achieved within the framework of heterotic M-theory whose effective action is only known to order $\kappa^{2/3}$. Let us therefore briefly indicate where and in which way features of heterotic M-theory will enter our analysis. By imposing supersymmetry, we are going to solve the gravitino Killing-spinor equation of M-theory. The heterotic M-theory characteristics enter on the one hand through specific G -fluxes originating from boundary or M5-brane sources and on the other hand through the chirality condition $\Gamma^{11}\eta = \eta$ on the susy-variation Majorana-parameter η . The important point is that *the information which is restricted to order $\kappa^{2/3}$ becomes only relevant if knowledge about the actual source strengths is required. However, to obtain the functional behaviour of $V(x^{11})$ this knowledge is not needed.* It suffices to assume that in the full heterotic M-theory the relevant sources can be localized in the x^{11} direction (as suggested by the anomaly considerations of [12] leading to the two E_8 gauge groups), i.e. they appear as $dG = \delta(x^{11} - z)S(x^m) \wedge dx^{11}$ in the Bianchi-identity. Thus, we will be able to answer the first question posed above. The actual value (and thereby the complete knowledge about heterotic M-theory beyond $\kappa^{2/3}$ order) for the 4-form source strength S only becomes indispensable if e.g. questions about the precise value of a zero or a minimum of $V(x^{11})$ should be answered. This is necessary if one wants to quantify the resulting value for Newton's Constant.

The main results of this paper concerning the questions posed above are

- The full dependence of the Calabi-Yau volume on the orbifold coordinate is quadratic with a manifest non-negative volume throughout the whole interval. The minimum of the parabola is a zero which can now be lifted by quantum effects.
- This zero of the nonlinear analysis corrects the zero of the first order analysis by a factor two. The critical lower-bound on Newton's Constant in the first order approximation is thereby further decreased by a factor $2/3$. This ameliorates the slight discrepancy between the lower bound on Newton's Constant and its observed value.
- The distance d between the walls cannot be stabilized by means of an effective potential, which is obtained by integrating out the internal dimensions of the eleven-dimensional heterotic M-theory action on the warped gravitational background. We demonstrate this explicitly at the $\kappa^{2/3}$ level by showing the vanishing of the four-dimensional cosmological constant.
- In the case where the volume does not depend on the orbifold coordinate one is able to reproduce the relation between warp-factor and torsion of the weakly coupled heterotic string.

2 Deriving the Full Relation between Warped Geometry and G -Flux in Heterotic M-Theory

Let us consider heterotic M-theory compactified on $CY_3 \times S^1/\mathbb{Z}_2$ with four external coordinates x^μ ; $\mu, \nu, \rho, \dots = 1, 2, 3, 4$ and seven internal coordinates x^u ; $u, v, w, x, y, z = 5, \dots, 11$. In the absence of any G -flux (for heterotic M-theory this amounts to considering only the leading order which is M-theory itself without boundary or M5-brane sources) the metric solution to the Killing-spinor equation, which describes a supersymmetry-preserving vacuum, is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{uv}(x^w) dx^u dx^v , \quad (2.2)$$

where g_{uv} decomposes into a direct product of the Calabi-Yau metric $g_{a\bar{b}}$ and the metric $g_{11,11}$ of the eleventh dimension. Without loss of generality one can set $g_{11,11} = 1$. We will denote the six real Calabi-Yau indices l, m, n, p, q, \dots while $\bar{l}, \bar{m}, \bar{n}, \bar{p}, \bar{q}, \dots$ are the

respective flat tangent space indices. The alternative choice of holomorphic and anti-holomorphic indices will be denoted $a, b, c \dots$ and $\bar{a}, \bar{b}, \bar{c} \dots$. Genuinely we have to take the boundary sources into account which require turning on a G -flux in the internal directions. This necessitates a more general metric, for which we choose the warp-factor Ansatz

$$\begin{aligned} ds^2 &= \hat{g}_{MN} dx^M dx^N ; \quad M, N = 1, \dots, 11 \\ &= e^{b(x^n, x^{11})} \eta_{\mu\nu} dx^\mu dx^\nu + e^{f(x^n, x^{11})} g_{lm}(x^n) dx^l dx^m + e^{k(x^n, x^{11})} dx^{11} dx^{11} . \end{aligned} \quad (2.3)$$

It will turn out that the appropriate G -flux of the relevant sources can be accommodated with this Ansatz. The most general Ansatz which allows for arbitrary G -flux compatible with supersymmetry will be considered in the last section.

The initial Calabi-Yau manifold possesses a closed Kähler-form $\omega_{a\bar{b}}$. However, a non-zero G -flux entails a non-trivial internal warp-factor e^f , thereby rendering the “deformed” Kähler-form $\hat{\omega}_{a\bar{b}} = e^f \omega_{a\bar{b}}$ non-closed. In this respect, the warp-factor e^f serves as a measure for the deviation from Kählerness of the internal complex threefold.

To preserve 4-dimensional Poincaré-invariance, we set all components of G with at least one external index to zero³. An important point is that in order to preserve supersymmetry the magnetic sources on the right-hand-side of the Bianchi-identity must be (2,2,1) forms [8]. I.e. they are forms with two holomorphic, two anti-holomorphic indices and one x^{11} -index. This is clear for the boundary sources and amounts for the M5-brane sources to an orientation parallel to the boundaries. Solving the Bianchi-identity, we see from the fact that the sources are (2,2,1) forms, that only the components $G_{\bar{a}\bar{b}c\bar{d}}$, $G_{ab\bar{c}11}$, $G_{\bar{a}\bar{b}\bar{c}11}$ can become non-zero.

2.1 The Killing-Spinor Equation

The supersymmetry-variation of the gravitino in low-energy M-theory is given in the full metric (2.3) by

$$\delta\Psi_I = \hat{D}_I \tilde{\eta} + \frac{\sqrt{2}}{288} \left(\hat{\Gamma}_{IJKLM} - 8\hat{g}_{IJ}\hat{\Gamma}_{KLM} \right) G^{JKLM} \tilde{\eta} , \quad (2.4)$$

where $\tilde{\eta} = e^{-\psi(x^m, x^{11})} \eta$. Here, η is the original covariantly constant spinor and the exponential-factor accounts for the correction if G -flux is turned on. We will assume ψ to

³An external $G_{1234} = \epsilon_{1234} \lambda(x^m, x^{11})$ would be compatible with Poincaré-symmetry but the known sources (boundaries, M5-branes) do not give rise to such a flux. Moreover, compatibility with the Bianchi-identity allows only a constant G_{1234} . Though such a sourceless constant field-strength is allowed by the non-compact external spacetime, we will set it to zero subsequently.

be real and see later on that this is indeed compatible with supersymmetry in the warped background. Subsequently, indices are raised and lowered with the full metric \hat{g}_{MN} , which is also how contractions are performed in (2.4). Setting the variation to zero in order to obtain a supersymmetry preserving solution, we obtain the Killing-spinor equation.

Covariant Derivative Contribution

Let us first deal with the part containing the covariant derivative of the Majorana-spinor $\tilde{\eta}$. Using the definition of the spin-connection for the warped-metric⁴

$$\begin{aligned}\Omega_{I\bar{J}\bar{K}}(\hat{e}) &= \frac{1}{2}(\hat{e}_{\bar{J}}^J \tilde{\Omega}_{IJ\bar{K}}(\hat{e}) - \hat{e}_{\bar{K}}^K \tilde{\Omega}_{IK\bar{J}}(\hat{e}) - \hat{e}_{\bar{J}}^J \hat{e}_{\bar{K}}^K \hat{e}_{\bar{I}}^{\bar{I}} \tilde{\Omega}_{JK\bar{I}}(\hat{e})) , \\ \tilde{\Omega}_{IJ\bar{K}}(\hat{e}) &= \partial_I \hat{e}_{\bar{K}J} - \partial_J \hat{e}_{\bar{K}I} ,\end{aligned}\tag{2.5}$$

allows to express the warped spin-connection through the initial one

$$\begin{aligned}\Omega_{\mu\bar{\nu}\bar{l}}(\hat{e}) &= \frac{1}{2} \hat{e}_l^m \hat{e}_{\bar{\nu}\mu} \partial_m b , & \Omega_{\mu\bar{\nu}\bar{1}}(\hat{e}) &= \frac{1}{2} \hat{e}_{\bar{\nu}\mu} \hat{e}_{\bar{1}}^{11} \partial_{11} b \\ \Omega_{l\bar{m}\bar{n}}(\hat{e}) &= -\hat{e}_{[\bar{m}}^p \hat{e}_{\bar{n}]l} \partial_p f + \Omega_{l\bar{m}\bar{n}}(e) , & \Omega_{l\bar{m}\bar{1}}(\hat{e}) &= \frac{1}{2} \hat{e}_{\bar{m}l} \hat{e}_{\bar{1}}^{11} \partial_{11} f \\ \Omega_{11\bar{l}\bar{1}}(\hat{e}) &= -\frac{1}{2} \hat{e}_l^m \hat{e}_{\bar{1}1,11} \partial_m k ,\end{aligned}\tag{2.6}$$

and all other terms are zero. This is done to employ the covariant constancy $D_I \eta = (\partial_I + \frac{1}{4} \Omega_{I\bar{J}\bar{K}}(e) \Gamma^{\bar{J}\bar{K}}) \eta = 0$ of the initial spinor-parameter, which brings us to

$$\begin{aligned}dx^I \hat{D}_I \tilde{\eta} &= \left(-dx^u \partial_u \psi + \frac{1}{4} dx^\mu \left[\hat{\Gamma}_\mu^l \partial_l b + \hat{\Gamma}_\mu^{11} \partial_{11} b \right] + \frac{1}{4} dx^l \left[\hat{\Gamma}_l^m \partial_m f + \hat{\Gamma}_l^{11} \partial_{11} f \right] \right. \\ &\quad \left. + \frac{1}{4} dx^{11} \hat{\Gamma}_{11}^l \partial_l k \right) \tilde{\eta} .\end{aligned}\tag{2.7}$$

Let us now specify, that our internal space actually consists of a Calabi-Yau and a separate eleventh dimension. The positive chirality condition $\Gamma^{11} \eta = \eta$ on the original space translates into

$$\hat{\Gamma}^{11} \tilde{\eta} = e^{-k/2} \tilde{\eta}\tag{2.8}$$

on the warped space. The condition that we have a covariantly constant spinor (and its complex conjugate) on the Calabi-Yau gives $\Gamma^a \eta = 0$, $\Gamma_{\bar{a}} \eta = 0$ and translates into $\hat{\Gamma}^a \tilde{\eta} = 0$, $\hat{\Gamma}_{\bar{a}} \tilde{\eta} = 0$. Using these relations plus the Dirac-algebra $\{\hat{\Gamma}^a, \hat{\Gamma}^{\bar{b}}\} = 2\hat{g}^{a\bar{b}}$, we end

⁴Here \bar{J}, \bar{K}, \dots denote flat 11-dimensional indices.

up with⁵

$$\begin{aligned}
dx^I \hat{D}_I \tilde{\eta} = & \left\{ \left[-\partial_a(\psi + \frac{f}{4})dx^a + \partial_{\bar{a}}(-\psi + \frac{f}{4})dx^{\bar{a}} - \partial_{11}\psi dx^{11} \right] \right. \\
& + \left[\frac{1}{4}e^{-k/2}\partial_{11}b dx_\mu \right] \hat{\Gamma}^\mu + \left[\frac{1}{4}e^{-k/2}\partial_{11}f dx_{\bar{a}} - \frac{1}{4}e^{-k/2}\partial_{\bar{a}}k dx_{11} \right] \hat{\Gamma}^{\bar{a}} \\
& \left. + \left[\frac{1}{4}\partial_{\bar{a}}b dx_\mu \right] \hat{\Gamma}^{\mu\bar{a}} + \left[\frac{1}{4}\partial_{\bar{b}}f dx_{\bar{a}} \right] \hat{\Gamma}^{\bar{a}\bar{b}} \right\} \tilde{\eta} .
\end{aligned} \tag{2.9}$$

G-Flux Contributions

Next, let us deal with the second term in the Killing equation, containing the G -flux. To obtain condense expressions, it proves convenient to parameterize the three sorts of allowed fluxes by defining

$$\alpha = \omega^{lm}\omega^{np}G_{lmnp} \tag{2.10}$$

$$\beta_l = \omega^{mn}G_{lmn11} \tag{2.11}$$

$$\Theta_{lm} = G_{lmnp}\omega^{np} . \tag{2.12}$$

where $\omega_{a\bar{b}} = -ig_{a\bar{b}}$, $\omega^{a\bar{b}} = ig^{a\bar{b}}$ denotes the Kähler-form of the initial Calabi-Yau manifold. The warped metric is related to the Kähler-form by $\hat{g}^{a\bar{b}} = -ie^{-f}\omega^{a\bar{b}}$. Subsequently, we will make use of

$$\begin{aligned}
\hat{g}^{a\bar{b}}\hat{g}^{c\bar{d}}G_{a\bar{b}c\bar{d}} &= -\frac{1}{4}e^{-2f}\alpha \\
\hat{g}^{b\bar{c}}G_{lb\bar{c}11} &= -\frac{i}{2}e^{-f}\beta_l \\
\hat{g}^{c\bar{d}}G_{lmc\bar{d}} &= -\frac{i}{2}e^{-f}\Theta_{lm}
\end{aligned} \tag{2.13}$$

to express the occuring contractions through the above defined parameters. In order to handle the various contractions of $\hat{\Gamma}$ -matrices with the G -flux, it is convenient to evaluate the expressions by first letting the matrices act on $\tilde{\eta}$ and employ $\hat{g}^{a11} = 0$, $\hat{\Gamma}^{11}\tilde{\eta} = e^{-k/2}\tilde{\eta}$, $\hat{\Gamma}^a\tilde{\eta} = 0$. Taking $\tilde{\eta}$ as the ground state, $\hat{\Gamma}^a$ and $\hat{\Gamma}^{\bar{a}}$ can be regarded as annihilation and

⁵It is interesting to look at the Killing-spinor equation with general internal G -flux but without applying the condition of \mathbb{Z}_2 invariance, i.e. $\Gamma^{11}\eta = \eta$, or using the Calabi-Yau condition $\Gamma^a\eta = 0$ (which means that we are considering an M-theory compactification on a smooth 7-manifold). In this case, which will be briefly treated in appendix A.1, the Killing-spinor equation can only be solved by trivial warp-factors and a vanishing G -flux. Hence, M-theory compactifications on smooth 7-manifolds neither allow for nontrivial warped metrics nor for nonzero internal G -flux. This agrees with the result of [2].

creation operators, respectively. This leads to some useful identities (A.112) collected in the appendix. With their help, we establish the various contractions (A.113) of the five-index $\hat{\Gamma}$ -matrices with the G -flux and also the contractions (A.114) of the three-index $\hat{\Gamma}$ -matrices with G . These can be found in the appendix, as well. Putting all this together, we arrive at the following expression for the second part of the Killing-spinor equation

$$\begin{aligned}
dx_I (\hat{\Gamma}^{IJKLM} - 8\hat{g}^{IJ}\hat{\Gamma}^{KLM}) G_{JKLM} \tilde{\eta} = & \left\{ 3e^{-k/2-f} [4i\beta_{\bar{a}} dx^{\bar{a}} + 12i\beta_a dx^a - e^{-f}\alpha dx_{11}] \right. \\
& - 3e^{-2f}\alpha dx_{\mu} \hat{\Gamma}^{\mu} + 3e^{-f} \left[-e^{-f}\alpha dx_{\bar{a}} + 12i\Theta_{\bar{a}}^{\bar{b}} dx_{\bar{b}} - 8i\beta_{\bar{a}} dx^{11} \right] \hat{\Gamma}^{\bar{a}} - 12ie^{-k/2-f}\beta_{\bar{a}} dx_{\mu} \hat{\Gamma}^{\mu\bar{a}} \\
& \left. + 3e^{-k/2} [4ie^{-f}\beta_{\bar{a}} dx_{\bar{b}} - 12G_{\bar{a}\bar{b}11} dx_{\bar{c}}] \hat{\Gamma}^{\bar{a}\bar{b}} \right\} \tilde{\eta} . \tag{2.14}
\end{aligned}$$

Complete Killing-Spinor Equation

Now, the complete Killing-spinor equation can be composed out of the two pieces (2.9) and (2.14) and is given by

$$\begin{aligned}
& dx^I \hat{D}_I \tilde{\eta} + \frac{\sqrt{2}}{288} dx^I \left(\hat{\Gamma}_{IJKLM} - 8\hat{g}_{IJ}\hat{\Gamma}_{KLM} \right) G^{JKLM} \tilde{\eta} \\
& = \left(\left[\left(-\partial_a \psi - \frac{1}{4}\partial_a f + i\frac{\sqrt{2}}{8}e^{-k/2-f}\beta_a \right) dx^a + \left(-\partial_{\bar{a}} \psi + \frac{1}{4}\partial_{\bar{a}} f + i\frac{\sqrt{2}}{24}e^{-k/2-f}\beta_{\bar{a}} \right) dx^{\bar{a}} \right. \right. \\
& \quad \left. \left. - \left(\frac{\sqrt{2}}{96}e^{k/2-2f}\alpha + \partial_{11}\psi \right) dx^{11} \right] + \frac{1}{4} \left[e^{-k/2}\partial_{11}b - \frac{\sqrt{2}}{24}e^{-2f}\alpha \right] dx_{\mu} \hat{\Gamma}^{\mu} + \frac{1}{4} \left[e^{-k/2}\partial_{11}f dx_{\bar{a}} \right. \right. \\
& \quad \left. \left. - \frac{\sqrt{2}}{24}e^{-2f}\alpha dx_{\bar{a}} + i\frac{1}{\sqrt{2}}e^{-f}\Theta_{\bar{a}}^{\bar{b}} dx_{\bar{b}} - \left(e^{-k/2}\partial_{\bar{a}}k + i\frac{\sqrt{2}}{3}e^{-f-k}\beta_{\bar{a}} \right) dx_{11} \right] \hat{\Gamma}^{\bar{a}} + \frac{1}{4} \left[\partial_{\bar{a}}b \right. \right. \\
& \quad \left. \left. - i\frac{\sqrt{2}}{6}e^{-k/2-f}\beta_{\bar{a}} \right] dx_{\mu} \hat{\Gamma}^{\mu\bar{a}} + \frac{1}{4} \left[\partial_{\bar{b}}f dx_{\bar{a}} + i\frac{\sqrt{2}}{6}e^{-k/2-f}\beta_{\bar{a}} dx_{\bar{b}} \right. \right. \\
& \quad \left. \left. - \frac{1}{\sqrt{2}}e^{-k/2}G_{\bar{a}\bar{b}11} dx_{\bar{c}} \right] \hat{\Gamma}^{\bar{a}\bar{b}} \right) \tilde{\eta} \\
& = 0 .
\end{aligned}$$

2.2 The Warp-Factor – Flux Relations

Setting the coefficients of the various $\hat{\Gamma}$ -matrices to zero, we have to distinguish carefully between the imaginary and the real part of the equations. For arbitrary vectors A_U, B_U ,

the sum $A^a B_a + A^{\bar{a}} B_{\bar{a}}$ is real, whereas the difference $A^a B_a - A^{\bar{a}} B_{\bar{a}}$ is purely imaginary. Furthermore α is a real parameter.

The $\hat{\Gamma}^\mu$ and $\hat{\Gamma}^{\mu\bar{a}}$ -Terms

From the terms proportional to the unit-matrix, $\hat{\Gamma}^\mu$ and $\hat{\Gamma}^{\mu\bar{a}}$ we receive the relations

$$8\partial_a\psi = \partial_a f = -2\partial_a b = i\frac{\sqrt{2}}{3}e^{-k/2-f}\beta_a \quad (2.15)$$

$$4\partial_{11}\psi = -\partial_{11}b = -\frac{\sqrt{2}}{24}e^{k/2-2f}\alpha. \quad (2.16)$$

The $\hat{\Gamma}^{\bar{a}}$ -Terms

The terms proportional to $\hat{\Gamma}^{\bar{a}}$ lead to

$$\partial_a k = i\frac{\sqrt{2}}{3}e^{-k/2-f}\beta_a, \quad (2.17)$$

which shows that the warp-factors f and k are equal up to an additive function F depending merely on x^{11}

$$f(x^W, x^{11}) = k(x^W, x^{11}) + F(x^{11}). \quad (2.18)$$

In the following we will set $F(x^{11})$ to zero since it can be eliminated by a simple reparameterization of x^{11} . Furthermore the $\hat{\Gamma}^{\bar{a}}$ terms yield the relation

$$\partial_{11}f = \frac{\sqrt{2}}{24}e^{k/2-2f}\alpha - i\frac{1}{\sqrt{2}}e^{k/2-f}\Theta^{\bar{a}}_{\bar{a}}, \quad \text{no sum over } \bar{a} \quad (2.19)$$

together with

$$\Theta^{\bar{b}}_{\bar{a}} = 0, \quad \bar{b} \neq \bar{a}. \quad (2.20)$$

Note that in (2.19) there is no summation over the antiholomorphic indices \bar{a} . Hence (2.19) implies the following isotropy-condition

$$\Theta^{\bar{1}}_{\bar{1}} = \dots = \Theta^{\bar{n}}_{\bar{n}}, \quad (2.21)$$

with n the complex dimension of the Calabi-Yau manifold. Using the identity $\sum_{\bar{a}=\bar{1}}^{\bar{n}} \Theta^{\bar{a}}_{\bar{a}} = -\frac{i}{2}e^{-f}\alpha$, it then follows that (2.19) simplifies to

$$\partial_{11}f = \frac{\sqrt{2}}{4} \left(\frac{1}{6} - \frac{1}{n} \right) e^{k/2-2f}\alpha. \quad (2.22)$$

Finally the $\hat{\Gamma}^{\bar{a}\bar{b}}$ terms lead to an equation, which can be simplified, using the relation for $\partial_a f$ from (2.15), to

$$ie^{-f}\beta_{[\bar{a}dx_{\bar{b}]}} = G_{\bar{a}\bar{b}11}^{\bar{c}}dx_{\bar{c}}. \quad (2.23)$$

The component of this equation with $\bar{c} \neq \bar{a}, \bar{b}$ leads to the following G -flux constraint

$$G_{\bar{a}\bar{b}11}^{\bar{c}} = 0 \quad , \quad \bar{c} \neq \bar{a}, \bar{b}, \quad (2.24)$$

whereas the $\bar{c} = \bar{a}$ and $\bar{c} = \bar{b}$ components simply reproduce the defining relation (2.11) for $\beta_{\bar{a}}$.

To summarize, the Killing-spinor equation leads to the set of equations (2.15), (2.16), (2.18), (2.22) together with the G -flux constraints (2.20), (2.21), (2.24).

We are now in a position to briefly check that our assumption of choosing ψ real does not lead to inconsistencies. For this purpose it is enough to show that $\text{Im } \psi$ is constant, which in particular means that a zero value can be maintained. Following [8], we use the above equation for $\partial_a \psi$ and obtain

$$\begin{aligned} \hat{g}^{a\bar{b}}\partial_a\partial_{\bar{b}}\text{Im}\psi &= \hat{g}^{a\bar{b}}\partial_a\partial_{\bar{b}}\left(\frac{\psi - \bar{\psi}}{2i}\right) = \frac{\sqrt{2}}{48}\hat{g}^{a\bar{b}}\left[\partial_{\bar{b}}(e^{-3f/2}\beta_a) + \partial_a(e^{-3f/2}\beta_{\bar{b}})\right] \\ &= \frac{\sqrt{2}}{48}e^{-3f/2}D^m\beta_m. \end{aligned}$$

Employing $D^m\beta_m = 0$, which can be obtained from the field equation for G , one establishes that $\text{Im } \psi$ is a harmonic function on a compact space and therefore has to be constant.

3 Implications of the Warped Geometry

Let us now analyze the above equations in more detail. Notice, that up to now we were not forced to specify whether we compactify on a CY_2 or a CY_3 – the complex dimension n of the CY_n entered as a free parameter.

Arbitrary G -flux parameters α, β are only compatible with a pure warp-factor description of the internal deformed Calabi-Yau in the 6-dimensional case with $n = 2$, as we will now see. For $n = 2$ we obtain

$$\partial_{11}f = -\frac{\sqrt{2}}{12}e^{k/2-2f}\alpha, \quad (3.25)$$

which says, together with (2.15),(2.16) that

$$8\partial_a\psi = \partial_af = -2\partial_ab \quad (3.26)$$

$$8\partial_{11}\psi = \partial_{11}f = -2\partial_{11}b \quad (3.27)$$

and implies $8\psi = f = k = -2b$. Here the warp-factors depend on both x^m and x^{11} . For $n \neq 2$ the $\partial_{11}f$ part receives a different prefactor and does not allow for this conclusion. Instead – as we will see explicitly for the case of $n = 3$ below – one has to set either (3.26) or (3.27) to zero to obtain a consistent solution.

The difference between the 4- and 6-dimensional cases was also pointed out in [8]. In contrast to CY_3 compactifications, a compactification on $K = K3 \times S^1/\mathbb{Z}_2$ to six dimensions allowed the inclusion of a general G -flux without the need to treat the Calabi-Yau coordinates and the orbifold coordinate differently. This led to a full derivation of the relation between warp-factors and G -flux without the need for a first order truncation. Starting on $M^{11} = \mathbf{R}^6 \times K$ with

$$ds^2 = e^b \eta_{\mu\nu} dx^\mu dx^\nu + e^f g_{uv} dx^u dx^v, \quad (3.28)$$

the equation of motion

$$D^u G_{uvwx} = 0 \quad (3.29)$$

is solved by the Ansatz (with the $\epsilon^{(0)}$ tensor in the original metric)

$$G_{uvwx} = -\epsilon_{uvwx}^{(0)} \partial^y w \quad (3.30)$$

together with the condition

$$\Delta^{(0)} w = \text{sources} \quad (3.31)$$

with the sources derived from the *sources* in the Bianchi equation $dG = \text{sources}$. The searched for connection between the warp factor and the G -flux then takes the form

$$\begin{aligned} e^b &= (c + 2\sqrt{2}w)^{-1/3} \\ e^f &= (c + 2\sqrt{2}w)^{2/3}. \end{aligned} \quad (3.32)$$

The fact that $e^f = (e^b)^{-2}$ which we also obtain from (3.26),(3.27) will again show up in the 4-dimensional case without dependence on x^{11} . The 6-dimensional metric is related via the decompactification limit to the 11-dimensional extreme M5-brane metric $ds^2 = \Delta^{1/3} \eta_{\mu\nu} dx^\mu dx^\nu + \Delta^{-2/3} \delta_{uv} dx^u dx^v$ where $\Delta^{-1} = c + 2\sqrt{2}w$ with $w = q/R^3$ and $R = \sqrt{x_u x^u}$ for G taken to be the magnetic field of a point charge at $x^A = 0$ of charge q .

But let us now turn to the 4-dimensional case. If we choose $n = 3$, we have

$$\partial_{11}f = -\frac{\sqrt{2}}{24}e^{k/2-2f}\alpha . \quad (3.33)$$

Taking mixed derivatives of f and b this implies that $\partial_a\partial_{11}f = 0$. A non-trivial solution is either obtained from $\partial_{11}f = 0$ or $\partial_af = 0$. The implications of these two cases will be analyzed in more detail in the following two sections.

4 Connection between Strong and Weak Coupling

In this section, we will present the connection to the heterotic string with torsion [3]. The choice, $\partial_{11}f = 0$, requires $\alpha = 0, \beta_a \neq 0$ and leads to

$$8\psi(x^m) = f(x^m) = k(x^m) = -2b(x^m) \quad (4.34)$$

without any dependence on x^{11} . The required sort of fluxes is obtained by solving the Bianchi-identity $dG = \sum_{i=1}^m \delta(x^{11} - z_i)S_i(x^m) \wedge dx^{11}$ with m sources by $G = \sum_{i=1}^m \delta(x^{11} - z_i)P_i(x^m) \wedge dx^{11}$ with $dP_i = S_i$. This type of geometry seems tailor-made for a smooth transition to the weakly coupled heterotic string, since any x^{11} dependence is lost. Indeed, we will now show that the heterotic M-theory relation between warp-factor and G -flux reproduces the corresponding relation (A.126) for the heterotic string with torsion.

The warp-factor belonging to the 4-dimensional external part multiplies the Minkowski-metric – both in the string and the M-theory case – and is therefore fixed in the sense that one does not have to take into account further coordinate-reparameterizations for a comparison. Let us therefore start with the relation between external warp-factor and G -flux by using (2.15) for ∂_ab plus $f = k = -2b$ and employing the definition of β_a to obtain

$$\partial_a(e^{-b}) = -\frac{\sqrt{2}}{3}G_{ab}{}^b{}_{11} . \quad (4.35)$$

The contraction on the right-hand-side is with respect to $\hat{g}^{b\bar{c}}$. To compare M-theory with string-theory [11] one has to perform an overall Weyl-rescaling involving the dilaton, $g_{MN}^\sigma = e^{2\phi/3}\hat{g}_{MN}$, which brings us to the string-frame. Here we have

$$\begin{aligned} \partial_a(e^{-b}) &= -\frac{\sqrt{2}}{3}e^{\frac{2\phi}{3}}G_{ab}{}^b{}_{11} , \\ ds^2 &= e^{b+\frac{2\phi}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + \dots . \end{aligned}$$

Finally, let us go over to the 10-dimensional Einstein-frame via $g_{AB}^E = e^{-\phi/2} g_{AB}^\sigma$ in which we obtained the heterotic string relation between warp-factor and torsion. We thus arrive at

$$\partial_a(e^{-b}) = -\frac{\sqrt{2}}{3} e^{\frac{\phi}{6}} G_{ab}{}^b{}_{11} , \quad (4.36)$$

$$ds^2 = e^{b+\frac{\phi}{6}} \eta_{\mu\nu} dx^\mu dx^\nu + \dots , \quad (4.37)$$

where again the contraction is performed with the metric of the actual frame, $(g^E)^{b\bar{c}}$. A comparison of the above metric with the heterotic string metric (A.117) shows that we have to identify 2ϕ with $b + \phi/6$, which gives

$$b = \frac{11}{6} \phi . \quad (4.38)$$

If we use this in (4.36), we receive the heterotic M-theory warp-factor – flux relation

$$\partial_a(e^{-2\phi}) = -\frac{4\sqrt{2}}{11} G_{ab}{}^b{}_{11} . \quad (4.39)$$

Setting $G_{ab}{}^b{}_{11}$ equal to $H_{ab}{}^b$ up to some constant normalization factor, we see that indeed the relation between warp-factor and torsion of the heterotic string (see appendix A.3 for relevant facts about the heterotic string with torsion and a derivation of the following formula in that context)

$$\partial_a(e^{-2\phi}) = -\frac{1}{2} H_{ab}{}^b . \quad (4.40)$$

can be reproduced from heterotic M-theory including G -flux. This represents a non-trivial check on the duality between the strongly and the weakly coupled heterotic regions in the presence of torsion.

The choice of fluxes treated in this subsection leads to a Calabi-Yau volume which does not depend on x^{11} . Moreover due to the deformation with the warp-factor e^f the Kähler-form is no longer closed (cf. eq. (A.122)). In addition the Ricci-tensor for the internal six-manifold becomes

$$R_{a\bar{b}}(e^f g_{mn}) = R_{a\bar{b}}(g_{mn}) + g_{a\bar{b}} g^{c\bar{d}} (2\partial_c f \partial_{\bar{d}} f + \partial_c \partial_{\bar{d}} f) - \partial_a f \partial_{\bar{b}} f + 2\partial_a \partial_{\bar{b}} f , \quad (4.41)$$

where the derivatives of f are determined through the G -flux by (2.15). Though $R_{a\bar{b}}(g_{mn}) = 0$ due to the Ricci-flatness of the initial Calabi-Yau space, we recognize that in the presence of G -flux the internal six-manifold also loses its property of being Ricci-flat.

5 The Analysis of the Warped Geometry in the Strong Coupling Case and Newton's Constant

The second choice, $\partial_a f = 0$, requires $\alpha \neq 0, \beta_a = 0$ and implies

$$4\psi(x^{11}) = f(x^{11}) = k(x^{11}) = -b(x^{11}) \quad (5.42)$$

without any x^m dependence. The necessary non-vanishing $G_{\bar{a}\bar{b}c\bar{d}}$ and vanishing $G_{\bar{a}\bar{b}c11}$ are obtained by solving the Bianchi-identity $dG = \sum_{i=1}^m \delta(x^{11} - z_i) S_i(x^m) \wedge dx^{11}$ through⁶ $G = \sum_{i=1}^m \Theta(x^{11} - z_i) S_i(x^m)$. Again $S_i(x^m)$ is a closed 4-form representing the strength of the i^{th} magnetic source.

Note that for $\alpha \neq 0, \beta_a = 0$ the internal six-manifold remains Kähler. This is due to the fact that the warp-factor f does not depend on x^m . Furthermore, we see from the general formula (4.41) for the Ricci-tensor that in this case the six-manifold also keeps its property of being Ricci-flat. In other words the six-manifold is still a Calabi-Yau space with volume depending on the “parameter” x^{11} .

5.1 The volume dependence on the orbifold direction

The volume of the Calabi-Yau, as measured by the warped metric, is given by $V(x^{11}) = \int d^6x \sqrt{\hat{g}_{CY_3}} = e^{3f} \int d^6x \sqrt{g_{CY_3}}$. The decisive part, which is responsible for the variation of the volume with x^{11} , is the factor e^{3f} . For its determination, we use $f = k$ and the equation for $\partial_{11} f$

$$\partial_{11}(e^{3f/2}) = -\frac{1}{8\sqrt{2}} \alpha \quad (5.43)$$

which is solved by

$$e^{3f(x^{11})/2} = e^{3f(0)/2} - \frac{1}{8\sqrt{2}} \int_0^{x^{11}} dz \alpha(z) . \quad (5.44)$$

Notice that α does not depend on the Calabi-Yau coordinates, which can be easily seen by acting with ∂_a on (5.43) and taking into account that $\partial_a f = 0$. Hence the variation of the Calabi-Yau volume with x^{11} is given by

$$V(x^{11}) = \left(1 - \frac{1}{2\sqrt{2}} \omega^{a\bar{b}} \omega^{c\bar{d}} \int_0^{x^{11}} dz G_{\bar{a}\bar{b}c\bar{d}}(x^m, z)\right)^2 V_1 , \quad (5.45)$$

where $V_1 = \int d^6x \sqrt{g_{CY_3}}$ is the Calabi-Yau volume in the initial metric. The integration constant $e^{3f(0)/2}$ has been set to 1 to obtain a smooth transition from $V(x^{11})$ to V_1 in

⁶The Heaviside step-function $\Theta(x)$ is defined by $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$.

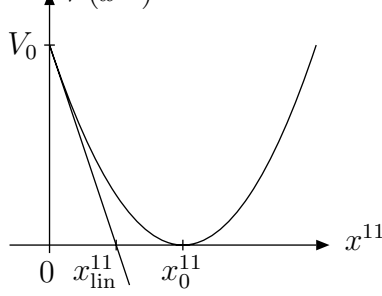


Figure 1: The quadratic dependence of the Calabi-Yau volume on the orbifold direction in the full geometry and its linear approximation to order $\kappa^{2/3}$. If higher order contributions are negligible then the linear approximation is valid for small x^{11} . The left boundary corresponds to the “visible” world.

case that we turn off any G -flux. The only assumption about the full heterotic M-theory that we will have to make is that the sources can still be localized at $x^{11} = z_i$ in the eleventh direction, i.e. that the Bianchi-identity possesses the form $dG = \sum_{i=1}^m \delta(x^{11} - z_i) S_i(x^m) \wedge dx^{11}$. Its solution $G = \sum_{i=1}^m \Theta(x^{11} - z_i) S_i(x^m)$ then leads to the following behaviour of the Calabi-Yau volume

$$V(x^{11}) = \left(1 - \sum_{i=1}^m (x^{11} - z_i) \Theta(x^{11} - z_i) \mathcal{S}_i\right)^2 V_1, \quad (5.46)$$

where $\mathcal{S}_i = \frac{1}{2\sqrt{2}} \omega^{a\bar{b}} \omega^{c\bar{d}} (S_i)_{a\bar{b}c\bar{d}}(x^m)$. Thus we get the remarkably simple result that *in the full treatment* the linear behaviour of the first order approximation gets replaced by a *quadratic behaviour*.

For the simplest case with only the two boundary sources at $z_1 = 0, z_2 = d$, we obtain $\alpha = 8\sqrt{2}\Theta(x^{11})\mathcal{S}_1$ with \mathcal{S}_1 representing the magnetic source of the “visible” boundary. This gives the warp-factor (beyond $x^{11} = x_0^{11} = 1/\mathcal{S}_1$, where the right-hand-side (rhs) becomes negative, we assume an analytic continuation of the left-hand-side through the rhs)

$$e^{3f(x^{11})/2} = 1 - \mathcal{S}_1 x^{11}, \quad (5.47)$$

which leads to the following volume dependence (see fig.1)

$$V(x^{11}) = (1 - \mathcal{S}_1 x^{11})^2 V_1. \quad (5.48)$$

Moreover – as becomes clear from the figure – with the quadratic volume behaviour *tiny quantum effects are now able to resolve the zero volume* as opposed to the linearized case (cf. in this respect also [9],[10]). We believe that the perfect square structure of the volume and thereby its zero at the minimum is related to supersymmetry. The reason

why $e^{3f/2}$ is linear in x^{11} can be traced back through the derivation of the warp-factor–flux relations to the fact, that we had imposed the condition (2.8). But this condition is nothing but the chirality condition $\Gamma^{11}\eta = \eta$ for the initial fermionic parameter η , which had to be imposed to preserve $N = 1$ supersymmetry in ten dimensions. On a more qualitative level one may argue that in general a solution of the Einstein equations is a solution to a second order differential equation with two integration constants. One gets a subclass of supersymmetry-preserving solutions by solving the Killing-Spinor equation instead. This special subclass exhibits in contrast only one integration constant. We suppose that the additional integration constant in the non-susy case parameterizes the ordinate position of the locus of minimal volume and becomes zero in the supersymmetric limit. This indicates that the zero could be lifted, not only by quantum corrections, but also classically by an appropriate Susy-breaking.

Here and in the following the right boundary will not be depicted – it would cut off the solution at some finite distance d . To determine the actual value of \mathcal{S}_1 (and thereby Newton’s Constant, see below) would require the actual knowledge of heterotic M-theory to all orders in $\kappa^{2/3}$. As we see, it is possible to parameterize our ignorance about these higher-order terms by \mathcal{S}_1 . In the phenomenological relevant case where $\mathcal{S}_1 > 0$ (a positive \mathcal{S}_1 is in accordance with the leading order approximation), a zero volume develops at $x_0^{11} = 1/\mathcal{S}_1$.

The x^{11} dependence of the full metric reads⁷

$$ds^2 = (1 - \mathcal{S}_1 x^{11})^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + (1 - \mathcal{S}_1 x^{11})^{2/3} (g_{lm}(x^n) dx^l dx^m + dx^{11} dx^{11}) \quad (5.50)$$

or in terms of the Calabi-Yau volume

$$ds^2 = \left(\frac{V(x^{11})}{V_1} \right)^{-1/3} \eta_{\mu\nu} dx^\mu dx^\nu + \left(\frac{V(x^{11})}{V_1} \right)^{1/3} (g_{lm}(x^n) dx^l dx^m + dx^{11} dx^{11}) .$$

⁷It is interesting to compare this solution to the domain-wall solution which arises in the effective 5-dimensional M-theory [9]. In the 11-dimensional decompactification limit it is given by (use eq. (4.19) of [9] and set $H_1 = H_2 = H_3 = H(y)$)

$$ds^2 = \frac{1}{H} \eta_{\mu\nu} dx^\mu dx^\nu + H (d\omega_1 + d\omega_2 + d\omega_3 + H dy^2) . \quad (5.49)$$

Here $d\omega_i$ are 2-dimensional line-elements spanning the internal six dimensions. Comparing both external 4-dimensional parts gives $H \sim V^{1/3}$, while from a comparison of the eleventh coordinate parts we gain the reparameterization $\sqrt{H} dy = dx^{11}$. Noting that $H \sim y$ [9], we obtain $x^{11} \sim y^{3/2}$ and thus $H \sim (x^{11})^{2/3}$. Therefore (5.50) and (5.49) show the same x^{11} dependence for both the external and the internal parts. The coordinate x^{11} which we are using instead of say y is distinguished by simple Bianchi-identities of the form $dG = \delta(x^{11} - z_i) S(x^m) \wedge dx^{11}$.

5.2 Extracting the First-Order Results

Finally, one would like to see the transition of the full expression for $V(x^{11})$ to the linearized expression which was derived in [8]. For this purpose one has to expand the sources into a power-expansion in $\kappa^{2/3}$. If we are only interested in sources coming from the boundary, then we know that they start at the order $\kappa^{2/3}$

$$\mathcal{S}_1 = \mathcal{S}_1^{(1)} \kappa^{2/3} + \mathcal{S}_1^{(2)} \kappa^{4/3} + \dots \quad (5.51)$$

For the first order approximation, we have to truncate this series after the first term, which indeed gives rise to a linear volume dependence

$$V(x^{11}) = \left(1 - 2\mathcal{S}_1^{(1)} \kappa^{2/3} x^{11}\right) V_1 + \mathcal{O}(\kappa^{4/3}) \quad (5.52)$$

as found in [8]. If we now read off the zero of $V(x^{11})$, we get $x_{\text{lin}}^{11} = 1/(2\mathcal{S}_1^{(1)} \kappa^{2/3})$ in the linearized case, while the full solution gives a different first order zero

$$x_0^{11} = 1/(\mathcal{S}_1^{(1)} \kappa^{2/3}) + \mathcal{O}(\kappa^{4/3}) \quad (5.53)$$

This little puzzle is resolved by noticing that the linear approximation (5.52) holds true only as long as $\mathcal{S}_1^{(1)} \kappa^{2/3} x^{11} \ll 1$ (plus similar conditions for the higher $\mathcal{S}_1^{(i)}$, $i \geq 2$ contributions). Because at the position of the zero, we face $\mathcal{S}_1^{(1)} \kappa^{2/3} x_{\text{lin}}^{11} \approx \mathcal{S}_1^{(1)} \kappa^{2/3} x_0^{11} = 1$, the linear approximation (5.52) breaks down and cannot be used to determine reliably the zero of $V(x^{11})$. Therefore, in contrast to the first order analysis, the actual zero at the first order level becomes larger by a factor 2

$$x_0^{11} = 2x_{\text{lin}}^{11} \quad (5.54)$$

This shows that if we place the “hidden” boundary not very close (such that $\mathcal{S}_1^{(1)} \kappa^{2/3} d$ is not much smaller than 1) to the “visible” boundary, we are forced to take into account higher order terms in $\kappa^{2/3}$. In particular this applies for the phenomenologically interesting region around the volume zero.

5.3 Inclusion of M5-branes

Let us briefly consider the case with three sources – two from the boundaries $\mathcal{S}_1, \mathcal{S}_2$ plus a further one \mathcal{S}_{M5} from an M5-brane placed in between at z_{M5} . With $\alpha = 8\sqrt{2}[\Theta(x^{11})\mathcal{S}_1 + \Theta(x^{11} - z_{M5})\mathcal{S}_{M5}]$ we get a warp-factor

$$e^{3f(x^{11})/2} = 1 - x^{11}\mathcal{S}_1 - (x^{11} - z_{M5})\Theta(x^{11} - z_{M5})\mathcal{S}_{M5} \quad (5.55)$$

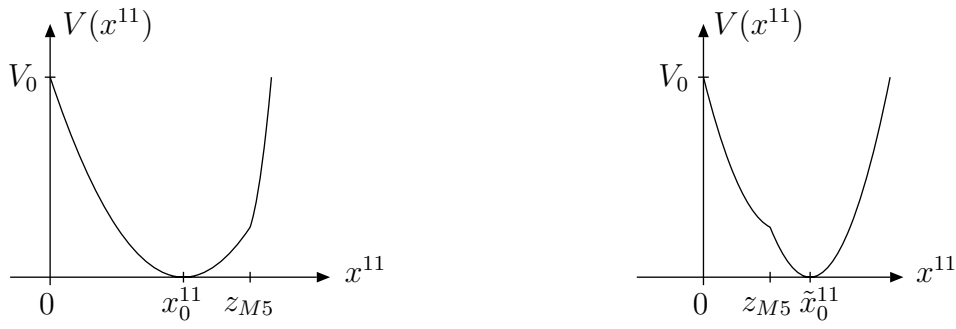


Figure 2: The figure shows the volume dependence in the presence of an additional M5-brane at z_{M5} , where we assume a positive M5-brane source \mathcal{S}_{M5} . In a) the situation for $x_0^{11} < \tilde{x}_0^{11} < z_{M5}$ is depicted while b) shows the behaviour for $x_0^{11} > \tilde{x}_0^{11} > z_{M5}$. The second boundary at $x^{11} = d$ is not depicted – it would truncate the solution at the finite distance d .

and the following volume dependence (see fig.2)

$$V(x^{11}) = \begin{cases} (1 - \mathcal{S}_1 x^{11})^2 V_0 & ; \quad x^{11} < z_{M5} \\ (1 - (\mathcal{S}_1 + \mathcal{S}_{M5})x^{11} + \mathcal{S}_{M5}z_{M5})^2 V_0 & ; \quad x^{11} \geq z_{M5} \end{cases} \quad (5.56)$$

The zero of the parabola for $x^{11} \geq z_{M5}$ lies at $\tilde{x}_0^{11} = (1 + \mathcal{S}_{M5}z_{M5})/(\mathcal{S}_1 + \mathcal{S}_{M5})$. Thus we see that an additional M5-brane will increase or decrease the slope of the volume parabola depending on whether it contributes a positive or negative \mathcal{S}_{M5} .

5.4 Newton's Constant

Let us briefly recall the evaluation of Newton's Constant in the first order analysis [8]. It is obtained by a dimensional reduction procedure together with an average Calabi-Yau volume $\langle V \rangle = \frac{1}{d} \int_0^d dx^{11} V(x^{11})$ through (1.1). It had been found [8] that

$$G_N \geq G_N^{crit,lin} \equiv \frac{\kappa^2}{8\pi V_1 x_{lin}^{11}} = \frac{\alpha_1^2}{16\pi^2} \left| \int_{CY_3} \omega \wedge \left(\frac{\text{tr} F^2 - \frac{1}{2} \text{tr} R^2}{8\pi^2} \right) \right|. \quad (5.57)$$

An approximation of the integral by $1/M_{GUT}^2$ delivers (with $M_{GUT} = 2 \times 10^{16} \text{GeV}$ and $\alpha_1 = 1/25$) the lower bound

$$G_N^{crit,lin} \simeq \frac{1}{(6.3 \times 10^{18} \text{GeV})^2}, \quad (5.58)$$

which is only slightly bigger than the actual phenomenological value $G_N^{phen} = 1/(1.3 \times 10^{19} \text{GeV})^2$.

In order to obtain effective 4-dimensional entities like G_N or the “hidden” gauge-coupling α_2 beyond the linear approximation, we have to keep the full x^{11} dependence of the fields and integrate out the seven compact dimensions (in contrast to performing a simple dimensional reduction). Starting with the 11-dimensional Einstein-Hilbert term

$$S = -\frac{1}{2\kappa^2} \int d^4x \int d^6x \int dx^{11} \sqrt{\hat{g}^{(4)}} \sqrt{\hat{g}_{CY_3}} \sqrt{\hat{g}_{11}} R(x^{11}) \quad (5.59)$$

we can use the full metric information (5.50) to explicitly integrate over x^{11} and the Calabi-Yau coordinates (the integration over the Calabi-Yau coordinates will be contained in the volume $V(x^{11})$). To gain a non-vanishing 4-dimensional curvature scalar, we consider slight perturbations around the flat 4-dimensional spacetime, $\eta_{\mu\nu} \rightarrow g_{\mu\nu}^{(4)}$. Keeping only the 4-dimensional curvature-scalar part from the reduction of the 11-dimensional curvature scalar, $R(\hat{g}_{MN}) = e^{-b} R(g_{\mu\nu}^{(4)}) + \dots$, we match the effective 4-dimensional Einstein-Hilbert action

$$S^{(4)} = -\frac{1}{16\pi G_N} \int d^4x \sqrt{g^{(4)}} R(g_{\mu\nu}^{(4)}) \quad (5.60)$$

with the following effective Newton’s Constant (in the “downstairs” picture [13], which we are employing, an additional factor of two multiplying the integral over x^{11} has to be taken into account)

$$G_N = \frac{\kappa^2}{16\pi \int_0^d dx^{11} e^{b(x^{11})/2} V(x^{11})} . \quad (5.61)$$

Note that it is not only the volume of the internal 7-fold but also an additional warp-factor stemming from the external 4-dimensional metric which enters the expression for the effective G_N . Though the CY-volume is manifestly positive, the square-root of the warp-factor e^b becomes negative beyond x_0^{11} if we use the analytic continuation mentioned before (5.47). Notice, however, that it is only the square root of the warp-factors which is continued into the negative region. The warp-factors themselves stay positive under this continuation, as can be seen explicitly by comparing (2.3) with (5.50). Also physical entities like the Ricci-tensor or the Riemann curvature scalar are well-behaved under this continuation.

If we now insert the known warp-factor and volume dependences, we arrive at

$$G_N = \frac{2}{3} G_N^{crit} \frac{1}{\left[1 - \left(1 - \frac{d}{x_0^{11}}\right)^{8/3}\right]} \geq \frac{2}{3} G_N^{crit} , \quad (5.62)$$

where $G_N^{crit} = \frac{\kappa^2}{4\pi V_1 x_0^{11}}$ approaches the lower bound of the first order analysis $G_N^{crit,lin}$ if $x_0^{11} \rightarrow 2x_{lin}^{11}$. Notice that $\frac{2}{3} G_N^{crit}$ places a lower bound on Newton’s Constant which

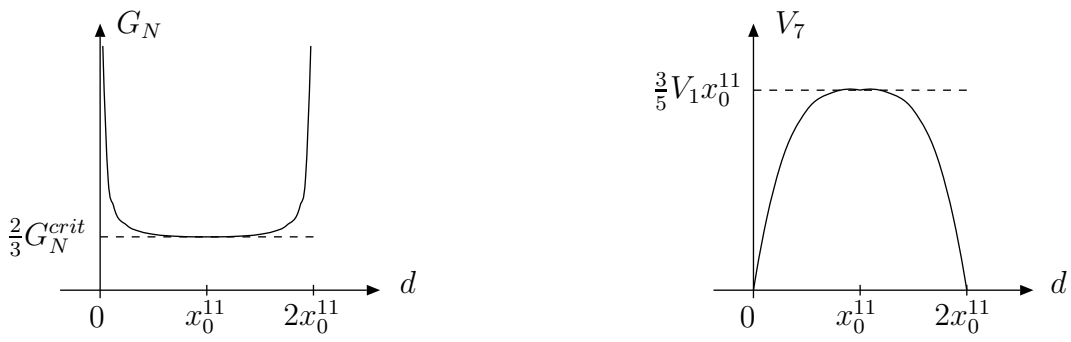


Figure 3: The dependence of G_N on d is shown in figure a). Qualitatively (modulo an external warp-factor) this can be understood from the corresponding variation of the seven-fold $CY_3 \times S^1/\mathbb{Z}_2$ volume $V(7\text{-fold})$ with d , figure b). Note that the decrease of $V(7\text{-fold})$ beyond x_0^{11} results from the analytic continuation of the warp-factors into the negative region.

depends via $x_0^{11} = 1/\mathcal{S}_1$ on the source strength. In the first order approximation this lowers the previously [8] obtained lower bound by a further factor $2/3$, which is welcomed for phenomenological purposes.

The dependence of G_N on d is symmetric around the zero-position x_0^{11} (see fig.3a). This implies that the effective Newton's Constant does not only diverge at $d = 0$ (as it already does at the zeroth-order approximation (1.1)) but also at $d = 2x_0^{11}$. Hence, the length of the orbifold-interval has to be upper-bounded by $d \leq 2x_0^{11}$. Notice furthermore, that qualitatively the inverse of the seven-fold $CY_3 \times S^1/\mathbb{Z}_2$ volume

$$\begin{aligned} V(7\text{-fold}) &= \int d^6x \, 2 \int_0^d dx^{11} \sqrt{\hat{g}_{CY_3}} \sqrt{\hat{g}_{11,11}} = 2 \int_0^d dx^{11} e^{k(x^{11})/2} V(x^{11}) \\ &= \frac{3}{5} V_1 x_0^{11} \left[1 - \left(1 - \frac{d}{x_0^{11}} \right)^{10/3} \right] \end{aligned} \quad (5.63)$$

reflects the main features of G_N (see fig.3b), as one would expect from the zeroth-order formula (1.1). However, quantitatively they differ by the contribution of an additional external warp-factor e^b which appears under the x^{11} integral of G_N .

Both the Calabi-Yau volume and the warp-factors are symmetric with respect to x_0^{11} . Therefore it is clear that this property also holds for the gauge-coupling (as a function of d) of the hidden boundary. Let us briefly derive the corresponding expression for the hidden gauge-coupling. Starting with the 11-dimensional gauge-kinetic term

$$S = -\frac{1}{8\pi(4\pi\kappa^2)^{2/3}} \int d^{10}x \sqrt{\hat{g}^{(4)}} \sqrt{\hat{g}_{CY_3}} \hat{g}^{AC} \hat{g}^{BD} F_{AB}^a F_{CD}^a \quad (5.64)$$

we focus on the 4-dimensional $F_{\mu\nu}^a$ components, which are supposed to depend solely on x^μ . Using (5.50), one obtains

$$S = -\frac{V_2}{8\pi(4\pi\kappa^2)^{2/3}} \int d^4x \sqrt{g^{(4)}} g^{(4)\mu\rho} g^{(4)\nu\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a, \quad (5.65)$$

where again we denote $V_2 = V(d)$. The fact that the warp-factor contributions arising from the 4-dimensional part of the full metric cancel each other is special to four dimensions and leads to the simple 4-dimensional hidden gauge-coupling

$$\alpha_2 = \frac{(4\pi\kappa^2)^{2/3}}{2V_2}. \quad (5.66)$$

Notice that the formula is the same as for the dimensional reduction (1.1). But here, we have to take the full quadratic Calabi-Yau volume at hidden boundary position $x^{11} = d$ instead of the first order linear volume. Again, we see that the hidden gauge-theory becomes strongly coupled if the second boundary is placed in the vicinity of the zero of the Calabi-Yau volume.

6 The Cosmological Constant vanishes at $\kappa^{2/3}$ Order

We saw previously that for the case with fluxes $\alpha \neq 0, \beta_a = 0$ we gain a quadratic behaviour in x^{11} for the Calabi-Yau threefold volume. The value of its zero is closely linked to the value of the 4-dimensional Newton's Constant. Hence, it is important to ask whether such a phenomenologically favoured value for the modulus d could be stabilized by means of an effective 4-dimensional potential. One could attack this problem by means of the effective 4- or 5-dimensional action of heterotic M-theory derived in [14]. Here we are going to examine the resulting potential for d for the derived warped graviational background by integrating out the internal dimensions of the original 11-dimensional heterotic M-theory action. We will find, as one expects from supersymmetry arguments, that this potential vanishes to first order in $\kappa^{2/3}$. Thus, in agreement with the 4-dimensional Minkowski-solution in (5.50) the effective 4-dimensional cosmological constant is zero at this order. Unfortunately, the analysis at higher orders cannot be performed, yet, since it would presuppose the complete knowledge of the heterotic M-theory action at the appropriate orders, which is still lacking.

Let us start with the bosonic action of 11-dimensional heterotic M-theory [13], which

under the condition that only the G -flux component $G_{\bar{a}\bar{b}c\bar{d}}$ contributes reads ($i = 1, 2$)

$$\begin{aligned}
S &= S_{11} + S_{10}^{(1)} + S_{10}^{(2)} , \\
S_{11} &= \frac{1}{2\kappa^2} \int d^{11}x \sqrt{\hat{g}} \left[-\hat{R}(\hat{g}_{MN}) - \frac{1}{4} \hat{G}_{\bar{a}\bar{b}c\bar{d}} \hat{G}^{\bar{a}\bar{b}c\bar{d}} \right] , \\
S_{10}^{(1)} &= \frac{1}{2\kappa^2} \int d^{10}x \sqrt{\hat{g}^{(1)}} \mathcal{L}^{(1)} , \\
S_{10}^{(2)} &= \frac{1}{2\kappa^2} \int d^{10}x \sqrt{\hat{g}^{(2)}} \mathcal{L}^{(2)} ,
\end{aligned} \tag{6.67}$$

where the hats are meant to indicate that the full warped metric is used for determination or contractions. To leading order in $\kappa^{2/3}$ the bosonic part of the boundary actions is given by

$$\mathcal{L}^{(i)} = -\frac{1}{2\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \text{tr} F_{\bar{a}\bar{b}}^{(i)} F^{(i)\bar{a}\bar{b}} + \mathcal{O}(\kappa^{4/3}) , \tag{6.68}$$

where to leading order in $\kappa^{2/3}$ the hats over the field-strengths have been omitted since contractions must now be performed with the original “zero-order” metric.

The Measure-Factors

For the case with varying volume we found in the last section the warp-factors relation $f(x^{11}) = k(x^{11}) = -b(x^{11})$. This allows to express $\sqrt{\hat{g}} = e^{-3b/2} \sqrt{g_{CY_3}}$ in terms of the measure on the original Calabi-Yau threefold without warp-factors. The condition to preserve supersymmetry gave

$$\partial_{11}(e^{-3b/2}) = -\frac{\sqrt{2}}{16} \alpha ,$$

which together with $\alpha = 8\sqrt{2}\Theta(x^{11})\mathcal{S}_1$ (note that for the case under consideration, we have $\partial_a\alpha = 0$ which means \mathcal{S}_1 is a constant) leads to

$$e^{-3b(x^{11})/2} = 1 - \mathcal{S}_1 x^{11} ,$$

thus determining the measure-factor as

$$\sqrt{\hat{g}} = (1 - \mathcal{S}_1 x^{11}) \sqrt{g_{CY_3}} . \tag{6.69}$$

Analogously the boundary measures are given by $\sqrt{\hat{g}^{(1)}} = e^{-b(x^{11}=0,d)} \sqrt{g_{CY_3}}$ which leads to

$$\sqrt{\hat{g}^{(1)}} = \sqrt{g_{CY_3}} , \quad \sqrt{\hat{g}^{(2)}} = (1 - \mathcal{S}_1 d)^{2/3} \sqrt{g_{CY_3}} . \tag{6.70}$$

Next let us express the 11-dimensional curvature-scalar for the warp-metric

$$\hat{R}(\hat{g}_{MN}) = \hat{g}^{KL} \partial^M \partial_M \hat{g}_{KL} - \partial^K \partial^L \hat{g}_{KL} + \hat{\Gamma}_{KL}^P \hat{\Gamma}_{MN}^Q \hat{g}_{PQ} (\hat{g}^{KL} \hat{g}^{MN} - \hat{g}^{KM} \hat{g}^{LN})$$

through the warp-factors b, f and the original Calabi-Yau curvature scalar. This gives

$$\begin{aligned} \hat{R} = e^{-k} & \left[\mathcal{D} \partial_{11}^2 b + \mathcal{N} \partial_{11}^2 f + \frac{\mathcal{D}(\mathcal{D}+1)}{4} (\partial_{11} b)^2 + \frac{\mathcal{N}(\mathcal{N}+1)}{4} (\partial_{11} f)^2 \right. \\ & \left. + \frac{\mathcal{D}\mathcal{N}}{2} \partial_{11} b \partial_{11} f - \frac{\mathcal{D}}{2} \partial_{11} b \partial_{11} k - \frac{\mathcal{N}}{2} \partial_{11} f \partial_{11} k \right] + e^{-f} R(g_{mn}) , \end{aligned}$$

where \mathcal{D} represents the real dimension of the non-compact external spacetime, while $\mathcal{N} = 2n$ denotes the real dimension of the complex n -dimensional internal Calabi-Yau manifold. For our concrete case with $\mathcal{D} = 4$, $n = 3$ and $f = k = -b$ plus a Ricci-flat Calabi-Yau manifold, we arrive at

$$\hat{R} = e^b \left[-2\partial_{11}^2 b + \frac{5}{2} (\partial_{11} b)^2 \right] .$$

Using $e^{-3b/2} = 1 - \mathcal{S}_1 x^{11}$ plus the \mathbb{Z}_2 symmetry of the orbifold (which introduces a jump in the first derivative of the metric at the orbifold fixed-points), we finally obtain

$$\hat{R} = -\frac{2}{9} \frac{\mathcal{S}_1^2}{(1 - \mathcal{S}_1 x^{11})^{8/3}} + \frac{8}{3} \mathcal{S}_1 \left(-\delta(x^{11}) + \frac{\delta(x^{11} - d)}{(1 - \mathcal{S}_1 d)^{5/3}} \right) . \quad (6.71)$$

for the curvature.

The Field-Strengths

Let us deal next with the field-strengths appearing in the bulk and boundary action. Using Einstein's equation

$$\begin{aligned} \hat{R}_{MN} - \frac{1}{2} \hat{R} \hat{g}_{MN} = & -\frac{1}{24} \left(4 \hat{G}_{MKLP} \hat{G}_N^{KLP} - \frac{1}{2} \hat{g}_{MN} \hat{G}_{KLPQ} \hat{G}^{KLPQ} \right) \\ & - \left(\delta(x^{11}) T_{AB}^{(1)} + \delta(x^{11} - d) T_{AB}^{(2)} \right) \delta_M^A \delta_N^B \end{aligned} \quad (6.72)$$

with the boundary energy-momentum tensor

$$\begin{aligned} T_{AB}^{(i)} = & \frac{1}{\sqrt{g_{11,11}}} \left(\frac{1}{2} \hat{g}_{AB} \mathcal{L}^{(i)} - \frac{\partial \mathcal{L}^{(i)}}{\partial \hat{g}^{AB}} \right) \\ = & \frac{1}{\sqrt{g_{11,11}} 2\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \left(\text{tr} F_{AC}^{(i)} F_B^{(i)C} - \frac{1}{4} \hat{g}_{AB} \text{tr} F_{CD}^{(i)} F^{(i)CD} \right) + \mathcal{O}(\kappa^{4/3}) , \end{aligned} \quad (6.73)$$

we may substitute the G -flux kinetic term in the bulk action by the expression

$$-\frac{1}{4} \hat{G}_{\bar{a}\bar{b}\bar{c}\bar{d}} \hat{G}^{\bar{a}\bar{b}\bar{c}\bar{d}} = 3\hat{R} - \frac{2}{3} \left(\delta(x^{11}) T_A^{(1)A} + \delta(x^{11} - d) T_A^{(2)A} \right) . \quad (6.74)$$

Thus the tree-level action becomes

$$S = \frac{1}{2\kappa^2} \left\{ \int d^{11}x \sqrt{\hat{g}} 2\hat{R} + \frac{1}{3} \sum_{i=1,2} \int d^{10}x \sqrt{\hat{g}^{(i)}} \left(2\hat{g}^{(i)AB} \frac{\partial \mathcal{L}^{(i)}}{\partial \hat{g}^{(i)AB}} - 7\mathcal{L}^{(i)} \right) \right\}. \quad (6.75)$$

For the boundary contributions let us concentrate on terms with two metric contractions which take expectation values on the Calabi-Yau manifold. It is this class of terms which appear at $\kappa^{2/3}$ order to which we ultimately have to truncate our results. This amounts to an additional warp-factor $e^{-2f(x^{11}=0,d)} = 1/(1 - \mathcal{S}_1 x^{11})^{4/3}|_{x^{11}=0,d}$ from the metric contractions.

The Effective Potential

Putting everything together, we can now integrate over the internal dimensions, which can be done explicitly for the eleventh dimension⁸. We thus obtain for the effective tree-level action

$$S = \frac{1}{6\kappa^2} \int d^{10}x \sqrt{g_{CY_3}} \left\{ \sum_{i=1,2} \frac{1}{(1 - \mathcal{S}_1 d^{(i)})^{2/3}} \left(2g^{lm} \frac{\partial \mathcal{L}^{(i)}}{\partial g^{lm}} - 7\mathcal{L}^{(i)} \right) + 28\mathcal{S}_1 \left(-1 + \frac{1}{(1 - \mathcal{S}_1 d)^{2/3}} \right) \right\}, \quad (6.76)$$

where we have defined $d^{(1)} = 0, d^{(2)} = d$. At this point, since the complete higher order in $\kappa^{2/3}$ terms of the boundary actions $\mathcal{L}^{(i)}$ are still unknown, we have to truncate the action to the first non-trivial $\kappa^{2/3}$ order to proceed further. First for the boundary contributions, this truncation gives

$$2g^{lm} \frac{\partial \mathcal{L}^{(i)}}{\partial g^{lm}} - 7\mathcal{L}^{(i)} = \frac{3}{2\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \text{tr} F_{a\bar{b}}^{(i)} F^{(i)a\bar{b}}. \quad (6.77)$$

To proceed further we are going to derive a relationship between both $\text{tr} F^{(i)2}$ terms under the integral over the Calabi-Yau manifold. Let us start with the CY-integral

$$\int d^6x \sqrt{g_{CY_3}} \left[\sum_{i=1,2} \text{tr} F_{lm}^{(i)} F^{(i)lm} - \text{tr} R_{lm} R^{lm} \right], \quad (6.78)$$

and promote it to an integral over the $CY_3 \times S^1/\mathbb{Z}_2$ manifold (we are working now at $\kappa^{2/3}$ order, which means that the integral measure as well as the contractions and the Riemann curvature tensor have to be taken in the original unwarped geometry)

$$\int d^7x \sqrt{g_{CY_3}} \left[\sum_{i=1,2} \left(\text{tr} F_{lm}^{(i)} F^{(i)lm} - \frac{1}{2} \text{tr} R_{lm} R^{lm} \right) \delta(x^{11} - d^{(i)}) \right]. \quad (6.79)$$

⁸We work in the downstairs picture and employ $\int_{-d}^d dx^{11} = 2 \int_0^d dx^{11}$.

Exploiting the supersymmetry-condition resulting from the vanishing of the gaugino-variation, $\omega^{lm}F_{lm} = 0$, plus the Calabi-Yau SU(3) holonomy condition, $\omega^{lm}R_{lm} = 0$, we see that the integral is proportional to

$$\begin{aligned} & \int d^7x \sqrt{g_{CY_3}} \omega^{lm} \omega^{np} \left[\sum_{i=1,2} \left(\text{tr} F_{[lm}^{(i)} F_{np]}^{(i)} - \frac{1}{2} \text{tr} R_{[lm} R_{np]} \right) \delta(x^{11} - d^{(i)}) \right] \\ & \propto \int_{CY_3 \times \mathbb{Z}_2} \omega \wedge \left[\sum_{i=1,2} \left(\text{tr} F^{(i)} \wedge F^{(i)} - \frac{1}{2} \text{tr} R \wedge R \right) \delta(x^{11} - d^{(i)}) \right] \wedge dx^{11}, \end{aligned}$$

The Bianchi-identity at this order reads [13]

$$dG = -\frac{1}{2\sqrt{2}\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \sum_{i=1,2} \left(\text{tr} F^{(i)} \wedge F^{(i)} - \frac{1}{2} \text{tr} R \wedge R \right) \delta(x^{11} - d^{(i)}) \wedge dx^{11},$$

which says that (6.78) is simply proportional to

$$\int_{CY_3 \times \mathbb{Z}_2} \omega \wedge dG = \int_{CY_3 \times \mathbb{Z}_2} d(\omega \wedge G) = \int_{CY_3} [\omega \wedge G]_{x^{11}=0} - \int_{CY_3} [\omega \wedge G]_{x^{11}=d}$$

By the inverse operation as before we convert the last integral to

$$\int_{CY_3} \omega \wedge G \propto \int_{CY_3} d^6x \sqrt{g_{CY_3}} \omega^{lm} \omega^{np} G_{lmnp} \propto \int_{CY_3} d^6x \sqrt{g_{CY_3}} \alpha. \quad (6.80)$$

However, since this integral in general (hence in particular for our choice of fluxes) does not depend on x^{11} at $\kappa^{2/3}$ order [8], the integral is the same for both boundaries thus rendering (6.78) vanishing.

If we neglect the $\text{tr} R^2$ contributions in (6.78) for the low-energy boundary action because they are of higher order in derivatives, we then obtain the desired relation

$$\int_{CY_3} d^6x \sqrt{g_{CY_3}} \text{tr} F^{(1)2} = - \int_{CY_3} d^6x \sqrt{g_{CY_3}} \text{tr} F^{(2)2}. \quad (6.81)$$

Together with the $\kappa^{2/3}$ order value for the source (again we suppress the higher order in derivative Riemann-curvature term for the low-energy action)

$$\mathcal{S}_1 = \frac{1}{4\pi} \left(\frac{\kappa}{4\pi} \right)^{2/3} \text{tr} F_{ab}^{(1)} F^{(1)a\bar{b}} \quad (6.82)$$

this gives the following tree-level effective action

$$S = \left(-1 + \frac{1}{(1 - \mathcal{S}_1 d)^{2/3}} \right) \frac{11}{12\pi\kappa^2} \left(\frac{\kappa}{4\pi} \right)^{2/3} \int d^{10}x \sqrt{g_{CY_3}} \text{tr} F_{ab}^{(1)} F^{(1)a\bar{b}}. \quad (6.83)$$

Recognizing that the whole expression has to be truncated to $\kappa^{2/3}$ order, the expansion of the prefactor for $\mathcal{S}_1 d \ll 1$ simply results in $-1 + 1 = 0$, which shows the desired result, namely that at this order the (tree level) cosmological constant vanishes.

7 The Case with General Flux: Beyond Warped Geometries

If one wants to relax the constraint that either α or β_a is zero (which we adopted up to now), then one has to generalize the previous pure warped geometry to a geometry which exhibits a deviation from the Calabi-Yau metric and is not describable by a warp-factor alone. With such a generalized Ansatz

$$ds^2 = \hat{g}_{MN} dx^M dx^N \quad (7.84)$$

$$= e^{b(x^W, x^{11})} \eta_{\mu\nu} dx^\mu dx^\nu + [g_{lm}(x^n) + h_{lm}(x^n, x^{11})] dx^l dx^m + e^{k(x^W, x^{11})} dx^{11} dx^{11} ,$$

we will see that the inconsistency which arose for the warp-factor f if both α and β_a were present, disappears and instead leads to constraints on the internal spin-connection.

The CY_3 metric split entails a corresponding split for the internal Vielbein $\hat{e}^{\bar{l}}_m = e^{\bar{l}}_m(x^n) + f^{\bar{l}}_m(x^n, x^{11})$. Again, we will express the spin-connection through the one belonging to the initial metric

$$\Omega_{\mu\bar{\nu}\bar{l}}(\hat{e}) = \frac{1}{2} \hat{e}^{\bar{m}}_{\bar{l}} \hat{e}_{\bar{\nu}\mu} \partial_{\bar{m}} b , \quad \Omega_{\mu\bar{\nu}\bar{1}\bar{1}}(\hat{e}) = \frac{1}{2} \hat{e}_{\bar{\nu}\mu} \hat{e}^{\bar{1}\bar{1}}_{\bar{1}\bar{1}} \partial_{\bar{1}\bar{1}} b$$

$$\Omega_{l\bar{m}\bar{n}}(\hat{e}) = \Omega_{l\bar{m}\bar{n}}(e) + \Omega_{l\bar{m}\bar{n}}^{(d)}(e, f) , \quad \Omega_{l\bar{m}\bar{1}\bar{1}}(\hat{e}) = \frac{1}{2} \hat{e}^{\bar{1}\bar{1}}_{\bar{1}\bar{1}} (\partial_{\bar{1}\bar{1}} f_{\bar{m}l} + \hat{e}^{\bar{m}}_{\bar{m}} \hat{e}^{\bar{l}}_{\bar{l}} \partial_{\bar{1}\bar{1}} f_{\bar{l}m}) \quad (7.85)$$

$$\Omega_{11\bar{l}\bar{m}}(\hat{e}) = \hat{e}^{\bar{l}}_{\bar{l}} \partial_{[\bar{l}11]} f_{\bar{m}]l} , \quad \Omega_{11\bar{1}\bar{1}}(\hat{e}) = -\frac{1}{2} \hat{e}^{\bar{l}}_{\bar{l}} \hat{e}_{\bar{1}\bar{1},11} \partial_{\bar{l}} k ,$$

with all remaining terms vanishing. Now the deviation from the initial CY-geometry is characterised by $f^{\bar{l}}_m$ and $\Omega_{l\bar{m}\bar{n}}^{(d)}(e, f)$. Both go to zero if we turn off the G -fluxes. Employing again the covariant constancy $D_I \eta = (\partial_I + \frac{1}{4} \Omega_{I\bar{J}\bar{K}}(e) \Gamma^{\bar{J}\bar{K}}) \eta = 0$ of the original spinor-parameter, leads to

$$dx^I \hat{D}_I \tilde{\eta} = \left(-dx^u \partial_u \psi + \frac{1}{4} dx^\mu \left[\hat{\Gamma}_\mu^{\bar{l}} \partial_{\bar{l}} b + \hat{\Gamma}_\mu^{\bar{1}\bar{1}} \partial_{\bar{1}\bar{1}} b \right] + \frac{1}{4} dx^l \left[\Omega_{lmn}^{(d)} \hat{\Gamma}^{mn} \right. \right. \\ \left. \left. + 2\Omega_{lm11}(\hat{e}) \hat{\Gamma}^m \hat{\Gamma}^{11} \right] + \frac{1}{4} dx^{11} \left[\hat{\Gamma}_{11}^{\bar{l}} \partial_{\bar{l}} k + \Omega_{11lm}(\hat{e}) \hat{\Gamma}^{lm} \right] \right) \tilde{\eta} . \quad (7.86)$$

Again, specifying that our internal space consists of a Calabi-Yau times an interval, we employ $\hat{\Gamma}^{11} \tilde{\eta} = e^{-k/2} \tilde{\eta}$ and $\hat{\Gamma}^a \tilde{\eta} = 0$, $\hat{\Gamma}_{\bar{a}} \tilde{\eta} = 0$ plus the Dirac-algebra $\{\hat{\Gamma}^a, \hat{\Gamma}^{\bar{b}}\} = 2\hat{g}^{a\bar{b}}$ to

obtain

$$\begin{aligned}
dx^I \hat{D}_I \tilde{\eta} = & \left\{ \left[\left(\frac{1}{4} \Omega_{la}^{(d)a} - \frac{1}{4} \Omega_{l\bar{a}}^{(d)\bar{a}} - \partial_l \psi \right) dx^l + \left(\frac{1}{4} \Omega_{11a}^a(\hat{e}) - \frac{1}{4} \Omega_{11\bar{a}}^{\bar{a}}(\hat{e}) - \partial_{11} \psi \right) dx^{11} \right] \right. \\
& + \left[\frac{1}{4} e^{-k/2} \partial_{11} b dx_\mu \right] \hat{\Gamma}^\mu + \left[\frac{1}{2} e^{-k/2} \Omega_{l\bar{a}11}(\hat{e}) dx^l - \frac{1}{4} e^{-k/2} \partial_{\bar{a}} k dx_{11} \right] \hat{\Gamma}^{\bar{a}} \\
& \left. + \left[\frac{1}{4} \partial_{\bar{a}} b dx_\mu \right] \hat{\Gamma}^{\mu\bar{a}} + \left[\frac{1}{4} \Omega_{l\bar{a}\bar{b}}^{(d)} dx^l + \frac{1}{4} \Omega_{11\bar{a}\bar{b}}(\hat{e}) dx^{11} \right] \hat{\Gamma}^{\bar{a}\bar{b}} \right\} \tilde{\eta} .
\end{aligned} \tag{7.87}$$

For the second part of the Killing-equation which consists of the contractions of $\hat{\Gamma}$ -matrices with the G -flux, it will be convenient to use the following abbreviations

$$G = \hat{g}^{a\bar{b}} \hat{g}^{c\bar{d}} G_{a\bar{b}c\bar{d}} \tag{7.88}$$

$$G_m = \hat{g}^{b\bar{c}} G_{m\bar{b}c11} \tag{7.89}$$

$$G_{mn} = \hat{g}^{c\bar{d}} G_{mnc\bar{d}} . \tag{7.90}$$

In order to eventually extract the real and imaginary parts of the Killing-spinor equation, we have to know their behaviour under complex conjugation, which is given by

$$\overline{G} = G , \quad \overline{G_a} = -G_{\bar{a}} , \quad \overline{G_a^{\bar{a}}} = -G_{\bar{b}}^{\bar{a}} . \tag{7.91}$$

Analogously, to the treatment in the previous section, we then arrive at

$$\begin{aligned}
dx_I (\hat{\Gamma}^{IJKLM} - 8\hat{g}^{IJ} \hat{\Gamma}^{KLM}) G_{JKLM} \tilde{\eta} = & \left\{ 3e^{-k/2} [-24G_{\bar{a}} dx^{\bar{a}} - 8G_a dx^a + 4G dx_{11}] \right. \\
& + 12G dx_\mu \hat{\Gamma}^\mu + 12 \left[G dx_{\bar{a}} - 6G_{\bar{a}}^{\bar{b}} dx_{\bar{b}} + 4G_{\bar{a}} dx^{11} \right] \hat{\Gamma}^{\bar{a}} + 24e^{-k/2} G_{\bar{a}} dx_\mu \hat{\Gamma}^{\mu\bar{a}} \\
& \left. + 12e^{-k/2} [2G_{\bar{b}} dx_{\bar{a}} - 3G_{\bar{a}\bar{b}11}^{\bar{c}} dx_{\bar{c}}] \hat{\Gamma}^{\bar{a}\bar{b}} \right\} .
\end{aligned} \tag{7.92}$$

With (7.88) and (7.92) we are then able to decompose the complete Killing-equation (2.4) into its external, CY- and 11-components. Thus unbroken supersymmetry finally translates into the following constraints on the spin-connection

$$\Omega_{ab}^{(d)b} - \Omega_{ab}^{(d)\bar{b}} = \frac{2\sqrt{2}}{3} e^{-k/2} G_a \tag{7.93}$$

$$\Omega_{\bar{b}\bar{c}}^{(d)a} = 0 \tag{7.94}$$

$$\Omega_{ab}^{(d)c} = \frac{\sqrt{2}}{6} e^{-k/2} (G_d \delta_{ab}^{cd} + 3G_{ab11}^c) \tag{7.95}$$

$$\Omega_{11a}^a(\hat{e}) = \Omega_{11\bar{a}}^{\bar{a}}(\hat{e}) \tag{7.96}$$

$$\Omega_{\bar{a}11}^b(\hat{e}) = \Omega_{11ab}(\hat{e}) = 0 \tag{7.97}$$

$$\Omega_{a11}^b(\hat{e}) = \frac{\sqrt{2}}{12} e^{k/2} (G \delta_a^b - 6G_a^b) , \tag{7.98}$$

where $\delta_{ab}^{cd} = \delta_a^c \delta_b^d - \delta_b^c \delta_a^d$. Additionally, the solution to the Killing-spinor equation provides us with further equations, which determine the warp-factors and covariant-spinor deviation in terms of the G -flux parameters

$$\partial_a b = \frac{\sqrt{2}}{3} e^{-k/2} G_a \quad (7.99)$$

$$\partial_{11} b = -\frac{\sqrt{2}}{6} e^{k/2} G \quad (7.100)$$

$$\partial_a k = -\frac{2\sqrt{2}}{3} e^{-k/2} G_a \quad (7.101)$$

$$\partial_a \psi = -\frac{\sqrt{2}}{12} e^{-k/2} G_a \quad (7.102)$$

$$\partial_{11} \psi = \frac{\sqrt{2}}{24} e^{k/2} G . \quad (7.103)$$

Similarly to the last section we obtain

$$8\psi(x^m, x^{11}) = k(x^m, x^{11}) = -2b(x^m, x^{11}) , \quad (7.104)$$

but this time a dependence on both x^m and x^{11} is allowed. Note that this relation is in accordance with the result of the first order approximation derived in [8].

The relation $\Gamma_{uv}^u(\hat{g}) = \frac{1}{\sqrt{\hat{g}}} \partial_v \sqrt{\hat{g}}$ between the Christoffel-symbols and the metric determinant enables us to find

$$\partial_{11} \sqrt{\hat{g}_{CY_3}} = \sqrt{\hat{g}_{CY_3}} \left(\Gamma_{u11}^u(\hat{g}_{CY_3 \times I}) - \frac{1}{2} \partial_{11} k \right) . \quad (7.105)$$

Via the relation between the Christoffel-symbols and the spin-connection, $\hat{e}^{\bar{u}}_x \Gamma_{vw}^x = \partial_v \hat{e}^{\bar{u}}_w + \Omega_v^{\bar{u}}_{\bar{x}} \hat{e}^{\bar{x}}_w$, we get $\Gamma_{u11}^u(\hat{g}_{CY_3 \times I}) - \frac{1}{2} \partial_{11} k = \Omega_{u11}^u(\hat{e}_{CY_3 \times I})$ and thereby

$$\partial_{11} \sqrt{\hat{g}_{CY_3}} = \sqrt{\hat{g}_{CY_3}} \Omega_{l11}^l(\hat{e}) , \quad (7.106)$$

where we have used that $\Omega_{1111}^{11}(\hat{e}) = 0$. Together with the constraint on $\Omega_{a11}^b(\hat{e})$, which gives $\Omega_{a11}^a(\hat{e}) = \Omega_{\bar{a}11}^{\bar{a}}(\hat{e}) = \sqrt{2} \left(\frac{6+n}{12} \right) e^{k/2} G$ ($n = \dim_{\mathbb{C}} CY_n$), we obtain ultimately

$$\partial_{11} \ln \sqrt{\hat{g}_{CY_3}} = \sqrt{2} \left(\frac{6+n}{6} \right) e^{k/2} G . \quad (7.107)$$

Employing the equation for $\partial_{11} b$, we can integrate this equation to obtain the following expression for the Calabi-Yau dependence on x^{11} (with $n = 3$)

$$V(x^{11}) = \int_{CY_3} d^6 x \sqrt{\hat{g}_{CY_3}} = \int_{CY_3} d^6 x e^{-9b(x^{11}, x^m)} C(x^m) , \quad (7.108)$$

where $C(x^m)$ arose as an integration constant by integrating (7.107) over x^{11} . We see that now the specification of the sources simply by means of their location in the eleventh direction is not enough to determine $V(x^{11})$. This stems from the fact that G , which determines the warp-factor b contains contractions with $\hat{g}^{a\bar{b}}$ which itself is x^{11} dependent. Therefore the specification $G_{a\bar{b}c\bar{d}} \propto \Theta(x^{11} - z_i)$ does not fully determine the x^{11} behaviour of G . However, it is definitely true that a non-trivial $V(x^{11})$ requires $G \neq 0$ and therefore in view of (7.100) $G_{a\bar{b}c\bar{d}} \neq 0$.

Another interesting aspect of turning on both $G_{a\bar{b}c\bar{d}}$ and $G_{ab\bar{c}11}$ derives from the fact, that we saw above how the fluxes determine the internal spin-connection $\Omega(\hat{e})$. Now it is well-known that the spin-connection determines the holonomy-group \mathcal{H} of a manifold through the path-ordered exponential of $\Omega(\hat{e})$ around a closed curve γ

$$\mathcal{P}e^{\int_{\gamma} \Omega_m(\hat{e}) dx^m} \in \mathcal{H} . \quad (7.109)$$

This is an interesting further link between the physics of G -fluxes and the geometry of the compactification space. A complex 3-dimensional Kähler-manifold exhibits $U(3)$ holonomy. But we already saw above that turning on a G -flux in general ruins the closedness property of the Kähler-form – therefore the new deformed manifolds are no longer Kähler. This means that we do expect a more general holonomy than $U(3)$.

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A Appendix

A.1 M-Theory on a 7-Manifold

In this appendix let us consider as an aside the compactification of M-theory on a smooth 7-manifold. Our aim is to investigate which kind of internal G -flux can be turned on if at the same time supersymmetry shall be preserved and which sort of warp-factors will appear. Let us therefore start with the warp-Ansatz

$$ds^2 = \hat{g}_{MN} dx^M dx^N = e^{b(x^w)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{f(x^w)} g_{uv} dx^u dx^v, \quad (\text{A.110})$$

for an initial 7-manifold with metric g_{uv} . We will assume only an internal nonzero G -flux. To solve the equation of motion for G , we write $G = {}^* dU$, with U a 2-form and the Hodge-operation is taken with respect to the 7-manifold. In components this reads

$$G_{uvwx} = \frac{1}{4!} e^{-f/2} \hat{\epsilon}_{uvwxyz_1 z_2 z_3} \partial^{z_1} U^{z_2 z_3}, \quad (\text{A.111})$$

where $\hat{\epsilon}_{uvwxyz_1 z_2 z_3}$ is defined as a tensor (rather than a tensor-density) containing a factor $\sqrt{\det \hat{g}_{uv}}$. The effect of a nonzero G on the fermionic supersymmetry-variation parameter η will be considered by using $\tilde{\eta} = e^{-\psi(x^u)} \eta$ instead of the initial η . Plugging this into the Killing-spinor equation for the gravitino eventually leads to

$$\begin{aligned} & \left(-dx^u \partial_u \psi + \frac{1}{4} dx^\mu \hat{\Gamma}_{\mu u} \partial^u b + \frac{1}{4} dx^u \hat{\Gamma}_{uv} \partial^v f - \frac{\sqrt{2}}{48 \times 7^3} e^{-f/2} dx^\mu \hat{\Gamma}_\mu \hat{\Gamma}^{uvw} \partial_u U_{vw} \right. \\ & \quad \left. - \frac{\sqrt{2}}{48 \times 7^2} e^{-f/2} dx^u \hat{\Gamma}^{vw} \partial_{[u} U_{vw]} + \frac{\sqrt{2}}{6 \times 7^4} e^{-f/2} dx^u \hat{\Gamma}_u{}^{vw} \partial_v U_{wx} \right) \tilde{\eta} \\ & = 0 \end{aligned}$$

This equation is solved by setting the various coefficients in front of the independent Gamma-matrices to zero, which shows that the warp-factors b , f and the spinor-correction factor ψ all have to be trivially constant. In addition $\partial_{[u} U_{vw]}$ has to be zero, which amounts to a zero internal G -flux. Thus M-theory compactifications on smooth 7-manifolds which are supposed to preserve some supersymmetry do not allow for non-zero internal G -flux and require at the same time trivial warp-factors.

A.2 G-Flux Contractions

First, we present those identities, which are used in the evaluation of the Killing-spinor equation

$$\begin{aligned}
\hat{\Gamma}^{e\bar{a}b\bar{c}\bar{d}}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{\bar{e}a\bar{b}c\bar{d}}\tilde{\eta} &= [\hat{\Gamma}^{\bar{e}}(\hat{g}^{a\bar{b}}\hat{g}^{c\bar{d}} - \hat{g}^{a\bar{d}}\hat{g}^{c\bar{b}}) + \hat{\Gamma}^{\bar{b}}(\hat{g}^{a\bar{d}}\hat{g}^{c\bar{e}} - \hat{g}^{a\bar{e}}\hat{g}^{c\bar{d}}) + \hat{\Gamma}^{\bar{d}}(\hat{g}^{a\bar{e}}\hat{g}^{c\bar{b}} - \hat{g}^{a\bar{b}}\hat{g}^{c\bar{e}})]\tilde{\eta} \\
\hat{\Gamma}^{abc\bar{d}}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{\bar{a}b\bar{c}\bar{d}}\tilde{\eta} &= [\hat{g}^{a\bar{b}}\hat{g}^{c\bar{d}} - \hat{g}^{a\bar{d}}\hat{g}^{c\bar{b}}]\tilde{\eta} \\
\hat{\Gamma}^{\bar{a}b\bar{c}\bar{d}}\tilde{\eta} &= [\hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\bar{b}}\hat{g}^{c\bar{d}} - \hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\bar{d}}\hat{g}^{c\bar{b}} + \hat{\Gamma}^{\bar{b}}\hat{\Gamma}^{\bar{d}}\hat{g}^{c\bar{a}}]\tilde{\eta} \\
\hat{\Gamma}^{ab\bar{c}11}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{\bar{a}b\bar{c}11}\tilde{\eta} &= e^{-h/2}[\hat{\Gamma}^{\bar{a}}\hat{g}^{b\bar{c}} - \hat{\Gamma}^{\bar{c}}\hat{g}^{b\bar{a}}]\tilde{\eta} \\
\hat{\Gamma}^{ab\bar{c}}\tilde{\eta} &= 0 \\
\hat{\Gamma}^{\bar{a}b\bar{c}}\tilde{\eta} &= [\hat{g}^{a\bar{b}}\hat{\Gamma}^{\bar{c}} - \hat{g}^{a\bar{c}}\hat{\Gamma}^{\bar{b}}]\tilde{\eta} \\
\hat{\Gamma}^{a\bar{b}}\tilde{\eta} &= \hat{g}^{a\bar{b}}\tilde{\eta}.
\end{aligned} \tag{A.112}$$

With their help and the definitions (2.10),(2.11),(2.12), we arrive at the contractions

$$\begin{aligned}
\hat{\Gamma}^{\mu\nu\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= -3[e^{-2f}\alpha\hat{\Gamma}^{\mu} + 4ie^{-k/2-f}\beta_{\bar{a}}\hat{\Gamma}^{\mu}\hat{\Gamma}^{\bar{a}}]\tilde{\eta} \\
\hat{\Gamma}^{euv\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= -12ie^{-k/2-f}\beta^e\tilde{\eta} \\
\hat{\Gamma}^{\bar{e}uv\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= -3[e^{-2f}\alpha\hat{\Gamma}^{\bar{e}} - 4ie^{-f}\Theta_{\bar{a}}^{\bar{e}}\hat{\Gamma}^{\bar{a}} - 4ie^{-k/2-f}\beta^{\bar{e}} \\
&\quad + 4ie^{-k/2-f}\beta_{\bar{a}}\hat{\Gamma}^{\bar{e}}\hat{\Gamma}^{\bar{a}} + 4e^{-k/2}\hat{\Gamma}^{\bar{a}}\hat{\Gamma}^{\bar{b}}G_{\bar{a}\bar{b}11}^{\bar{e}}]\tilde{\eta} \\
\hat{\Gamma}^{11uv\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= -3e^{-k/2-2f}\alpha\tilde{\eta}
\end{aligned} \tag{A.113}$$

and

$$\begin{aligned}
\hat{g}^{\mu\nu}\hat{\Gamma}^{\nu\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= 0 \\
\hat{g}^{a\bar{u}}\hat{\Gamma}^{\nu\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= -3ie^{-k/2-f}\beta^a\tilde{\eta} \\
\hat{g}^{\bar{a}u}\hat{\Gamma}^{\nu\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= -3[ie^{-k/2-f}\beta^{\bar{a}} + ie^{-f}\Theta_{\bar{e}}^{\bar{a}}\hat{\Gamma}^{\bar{e}} - e^{-k/2}\hat{\Gamma}^{\bar{b}\bar{c}}G_{\bar{b}\bar{c}11}^{\bar{a}}]\tilde{\eta} \\
\hat{g}^{11u}\hat{\Gamma}^{\nu\nu\nu}G_{\nu\nu\nu\nu}\tilde{\eta} &= 3ie^{-f}\beta_{\bar{e}}\hat{\Gamma}^{\bar{e}}\hat{g}^{11,11}\tilde{\eta},
\end{aligned} \tag{A.114}$$

which are used in the main text.

A.3 The Heterotic String with Torsion

The Ansatz traditionally used [1] in compactifications of the 10-dimensional heterotic $E_8 \times E_8$ string on CY_3 to four dimensions with $\mathcal{N} = 1$ supersymmetry is to make the susy-variations of the gravitino ψ_M , the dilatino λ and the gluino χ^a ($i, j = 5, \dots, 10$)

$$\begin{aligned}\delta\psi_i &= \frac{1}{\kappa} D_i \eta + \frac{\kappa}{32g^2\phi} (\Gamma_i{}^{jkl} - 9\delta_i^j \Gamma^{kl}) \eta H_{jkl} \\ \delta\lambda &= -\frac{1}{\sqrt{2}\phi} (\Gamma \cdot \partial\phi) \eta + \frac{\kappa}{8\sqrt{2}g^2\phi} \Gamma^{ijk} \eta H_{ijk} \\ \delta\chi &= -\frac{1}{4g\sqrt{\phi}} \Gamma^{ij} F_{ij} \eta\end{aligned}\tag{A.115}$$

vanish by assuming that $H = d\phi = 0$. Here ϕ is the dilaton and H the gauge-invariant field strength of the NSNS 2-form B , which in addition has to fulfil the Bianchi identity

$$dH = \text{tr} R \wedge R - \frac{1}{30} \text{tr} F \wedge F .\tag{A.116}$$

This leads to the consequence that CY_3 is a Kähler manifold with $c_1(CY_3) = 0$ and $SU(3)$ holonomy (and the gauge field A being a holomorphic connection on a holomorphic vector bundle V over the Calabi-Yau threefold CY_3 obeying the Donaldson-Uhlenbeck-Yau equation).

This Ansatz was generalized in [3] to include a non-vanishing torsion $H \neq 0$ where solutions leading again to $\mathcal{N} = 1$ supersymmetry in $D=4$ were obtained by allowing for a warp-factor $e^{2D(y)}$ in the metric (in Einstein-frame)

$$g_{AB}^E(x, y) = e^{2D(y)} g_{AB}(x, y) = e^{2D(y)} \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{mn}(y) \end{pmatrix},\tag{A.117}$$

where we denote 10-dimensional indices by A, B, C, \dots . It turns out that D has to be the dilaton ϕ . The torsion and the dilaton are determined by

$$H = \frac{i}{2} (\bar{\partial} - \partial) J\tag{A.118}$$

$$e^{8\phi} = e^{8\phi_0} ||\Omega|| ,\tag{A.119}$$

where the fundamental (1,1) form J is built out of the complex structure $J_m{}^n$ as $J = \frac{1}{2} J_m{}^n g_{np} dy^m \wedge dy^p = i g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$ (in our conventions J equals up to a minus-sign the Kähler-form ω) and Ω is the (determined up to an overall constant) holomorphic 3-form with norm $||\Omega|| = (\Omega_{a_1 a_2 a_3} \bar{\Omega}_{\bar{b}_1 \bar{b}_2 \bar{b}_3} g^{a_1 \bar{b}_1} g^{a_2 \bar{b}_2} g^{a_3 \bar{b}_3})^{1/2}$. To recognize the relation between H (commonly called torsion) and the original torsion, we note that the metric torsion of a complex manifold is specified by the expression

$$T_{bc}^a = -2g^{\bar{d}a} g_{\bar{d}[b,c]}\tag{A.120}$$

and its complex conjugate. Hence, the above expression for H can be explicitly expressed through the metric torsion via

$$H = \frac{1}{4} \left(T_{a\bar{b}\bar{c}} dz^a \wedge dz^{\bar{b}} \wedge dz^{\bar{c}} + T_{\bar{c}ab} dz^a \wedge dz^b \wedge dz^{\bar{c}} \right) . \quad (\text{A.121})$$

Finally, the link between H and the warp-factor is given implicitly by the dilatino equation $\delta\lambda = 0$, which manifests itself in the following relationship

$$\begin{aligned} d^\dagger J &= i(\partial - \bar{\partial}) \ln ||\Omega|| \\ &= 8i(\partial - \bar{\partial})(\phi - \phi_0) . \end{aligned} \quad (\text{A.122})$$

From the left-hand side of this equation it can be easily discerned, that the right-hand side serves as a measure for the non-Kählerness of the compactification manifold. Therefore, by turning on H -torsion, the compactification manifold becomes deformed to a manifold which is no longer Kähler.

To gain a more explicit relation between the H -torsion and the resulting warp-factor, we note that the dilatino equation $\delta\lambda = 0$ can be alternatively written as [3]

$$8\partial_m \phi = J_m{}^n \nabla_p J_n{}^p . \quad (\text{A.123})$$

Here, the covariant derivative is constructed out of the initial metric g_{MN} without warp-factor. The H -covariant constancy of the complex structure [3]

$$\nabla_m J_n{}^p - H_{qm}{}^p J_n{}^q - H^q{}_{mn} J_q{}^p = 0 \quad (\text{A.124})$$

plus its property to square to minus the identity, $J_m{}^n J_n{}^p = -\delta_m^p$, serve together with $J_{a\bar{b}} = ig_{a\bar{b}}$ to derive

$$8\partial_a \phi = H_{ab}{}^b - H_{a\bar{b}}{}^{\bar{b}} . \quad (\text{A.125})$$

The contraction is with respect to the initial metric in whose frame the relation holds. Equation (A.118), which relates H -torsion with metric torsion, reads in components $H_{a\bar{b}\bar{c}} = -g_{\bar{c}[a,b]}$ and leads to $H_{a\bar{b}}{}^{\bar{b}} = -H_{ab}{}^b$. Finally, to obtain the relation between the warp-factor ϕ and H -torsion in the Einstein-frame, we have to transform the contractions according to the rescaling $g_{AB} = e^{-2\phi} g_{AB}^E$ from the initial frame to the Einstein-frame and gain

$$\partial_a (e^{-2\phi}) = -\frac{1}{2} H_{ab}{}^b . \quad (\text{A.126})$$

The contraction on the right-hand-side is now understood to be carried out with g_{AB}^E .

Furthermore warp-geometries appear in heterotic five-brane solutions preserving supersymmetry. They were obtained ([4],[5],[6]; cf. also the axionic instantons in [7]) with the Ansatz $(k, l, m, n = 7, \dots, 10)$

$$g_{mn} = e^{2\phi} \delta_{mn}$$

$$H_{mnl} = -\epsilon_{mnl}{}^k \partial_k \phi$$

showing again that turning on torsion leads to a warp factor.

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